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I.—ON THE PROOF OF THE PROPOSITION THAT $(Mx + Ny)^{-1}$ IS AN INTEGRATING FACTOR OF THE HOMOGENEOUS DIFFERENTIAL EQUATION $M + N\frac{dy}{dx} = 0$.

By G. G. STOKES, M.A. Fellow of Pembroke College.

A FALLACIOUS proof is sometimes given of this proposition, which ought to be examined. The substance of the proof is as follows.

Let us see whether it is possible to find a multiplier V, a homogeneous function of x and y, which shall render $Mdx + Ndy$ an exact differential. Let M and N be of n, and V of p dimensions; let

$$dU = V(Mdx + Ndy) \ldots\ldots\ldots (1);$$

then, on properly choosing the arbitrary constant in U, U will be a homogeneous function of $n + p + 1$ dimensions, } (A),

whence, by a known theorem,

$$(n + p + 1)\, U = x\frac{dU}{dx} + y\frac{dU}{dy} = V(Mx + Ny) \ldots (2);$$

therefore, dividing (1) by (2),

$$\frac{dU}{(n + p + 1)\, U} = \frac{Mdx + Ndy}{Mx + Ny};$$

and the first side of this equation being an exact differential, it follows that the second side is so also, and consequently that $(Mx + Ny)^{-1}$ is an integrating factor.

Now the factor so found is of $-n - 1$ dimensions; so that the first side of (2) is zero. In fact, we shall see that the statement (A) is not true as applied to the case in question, unless $Mx + Ny = 0$.

The general form of a function of x of n dimensions is Ax^n. The general form of a homogeneous function of x and y of n dimensions is $x^n \psi\left(\frac{y}{x}\right)$. The integral of the first is in general $\frac{Ax^{n+1}}{n+1}$, omitting the arbitrary constant; and consequently the dimensions of the function are increased by unity by integration. But in the particular case in which $n = -1$, the integral is $A \log x$, which is not a quantity of o dimensions, at least according to the definition just given, *according to which definition only* is the proposition with reference to homogeneous functions assumed in (2) true. Let us now examine in what cases U will be of $n+p+1$ dimensions.

Putting $M = M_0 x^n$, $N = N_0 x^n$, $y = xz$, M_0 and N_0 will be functions of z alone, and we shall have

$$Mdx + Ndy = x^n \{(M_0 + N_0 z)\, dx + N_0 x\, dz\}.$$

If $M_0 + N_0 z = 0$, i.e. if $Mx + Ny = 0$, we see that x^{-n-1} will be an integrating factor. The integral, being a function of z, will be of 0 dimensions, and both sides of (2) will be zero.

If $Mx + Ny$ is not equal to 0, we may multiply and divide by $(M_0 + N_0 z)\, x$, and we have

$$Mdx + Ndy = x^{n+1} (M_0 + N_0 z) \left(\frac{dx}{x} + \frac{N_0 dz}{M_0 + N_0 z}\right).$$

Hence we see that $\{x^{n+1} (M_0 + N_0 z)\}^{-1}$ or $(Mx + Ny)^{-1}$ is an integrating factor. For this factor we have

$$U = \log (x) + \phi\left(\frac{y}{x}\right),$$

ϕ denoting the function arising from the integration with respect to z.

In this case we have $x \frac{dU}{dx} + y \frac{dU}{dy} = 1$, not $= 0$.

It may be of some interest to enquire in what cases an exact differential of any number of independent variables, in which the differential coefficients are homogeneous functions of n dimensions, has an integral which is a homogeneous function of $n+1$ dimensions.

Let $dU = Mdx + Ndy + Pdz + \ldots$ be the exact differential. Let $y = y'x$, $z = z'x \ldots$, $M = M_0 x^n$, $N = N_0 x^n \ldots$, so that M_0, $N_0 \ldots$ are functions of y', $z' \ldots$ only; then

$$dU = x^n \{(M_0 + N_0 y' + P_0 z' \ldots)\, dx + (N_0 dy' + P_0 dz' \ldots)\, x\}.$$

First, suppose the coefficient of dx in this equation to be zero, or $Mx + Ny + Pz \ldots = 0$; then the expression for dU cannot be an exact differential unless $n = -1$. In this case U will be a function of y', $z' \ldots$, and will therefore be a homogeneous function of $n+1$ or 0 dimensions.

Secondly, suppose the coefficient of dx not to be zero; then

$$dU = x^{n+1}(M_0 + N_0 y' \ldots)\left(\frac{dx}{x} + \frac{N_0 dy' + P_0 dz' \ldots}{M_0 + N_0 y' + P_0 z' \ldots}\right)$$

$$= (Mx + Ny + Pz \ldots)\left(\frac{dx}{x} + \frac{N_0 dy' + P_0 dz' \ldots}{M_0 + N_0 y' + P_0 z' \ldots}\right) \ldots (3).$$

Now I say that $\dfrac{N_0 dy' + P_0 dz' \ldots}{M_0 + N_0 y' + P_0 z' \ldots}$ is the exact differential of a function of the independent variables y', $z' \ldots$, or, taking $y, z \ldots$ for the independent variables instead of $y', z' \ldots$, x being supposed constant, and putting for M_0, $N_0 \ldots$ their values, that $\dfrac{Ndy + Pdz + \ldots}{Mx + Ny + Pz \ldots}$ is an exact differential.

For, putting $Mx + Ny + Pz \ldots = D$, in order that the quantity considered should be an exact differential, it is necessary and sufficient that the system of equations of which the type is $\dfrac{d\frac{N}{D}}{dz} = \dfrac{d\frac{P}{D}}{dy}$ should be satisfied. This equation gives

$$D\left(\frac{dN}{dz} - \frac{dP}{dy}\right) + P\frac{dD}{dy} - N\frac{dD}{dz} = 0.$$

Now, since $\dfrac{dN}{dz} = \dfrac{dP}{dy}$, by the conditions of $Mdx + Ndy + Pdz \ldots$ being an exact differential, the above equation becomes $P\dfrac{dD}{dy} - N\dfrac{dD}{dz} = 0$, or

$$P\left(\frac{dM}{dy}x + \frac{dN}{dy}y + \frac{dP}{dy}z \ldots\right) - N\left(\frac{dM}{dz}x + \frac{dN}{dz}y + \frac{dP}{dz}z \ldots\right) = 0.$$

Replacing $\dfrac{dM}{dy}$, $\dfrac{dP}{dy} \ldots$ by $\dfrac{dN}{dx}$, $\dfrac{dN}{dz} \ldots$ and $\dfrac{dM}{dz}$, $\dfrac{dN}{dz} \ldots$ by $\dfrac{dP}{dx}$, $\dfrac{dP}{dy} \ldots$, this equation becomes

$$P\left(\frac{dN}{dx}x + \frac{dN}{dy}y + \frac{dN}{dz}z \ldots\right) - N\left(\frac{dP}{dx}x + \frac{dP}{dy}y + \frac{dP}{dz}z \ldots\right) = 0.$$

Now
$$\frac{dN}{dx}x + \frac{dN}{dy}y + \ldots = nN,$$
$$\frac{dP}{dx}x + \frac{dP}{dy}y + \ldots = nP,$$

and therefore the above equation *is* satisfied. Hence

$$\frac{Ndy + Pdz \ldots}{Mx + Ny + Pz \ldots}, \text{ or its equal } \frac{N_0 dy' + P_0 dz' \ldots}{M_0 + N_0 y' + P_0 z' \ldots},$$

is an exact differential $d\psi(y', z' \ldots)$. Consequently equation (3) becomes

$$dU = (Mx + Ny + Pz \ldots)\, d\{\log x + \psi(y', z' \ldots)\};$$

which equation being by hypothesis integrable, it follows that

$$Mx + Ny + Pz \ldots = \phi\{\log x + \psi(y', z' \ldots)\}:$$

and $Mx + Ny \ldots$ being moreover a homogeneous function of $n+1$ dimensions, it is clear that we must have $\phi(\alpha) = A e^{(n+1)\alpha}$. Hence we have

$$dU = A x^{n+1} e^{(n+1)\psi}\, d(\log x + \psi).$$

If now $n+1$ is not equal to 0, we have

$$U = \frac{A x^{n+1} e^{(n+1)\psi}}{n+1},$$

omitting the constant; but if $n = -1$, we have

$$U = A(\log x + \psi) + C.$$

We see then that if $Mx + Ny + Pz \ldots = 0$, which can only happen when $n = -1$, U will be a homogeneous function of $n+1$ or 0 dimensions. If $Mx + Ny + Pz \ldots\ldots$ is not equal to 0, then, if $n+1$ is not equal to 0, and the constant in U is properly chosen, U will be a homogeneous function of $n+1$ dimensions, but if $n+1=0$, U will not be a homogeneous function of 0 dimensions, but will contain $\log x$. Of course it might equally have contained the logarithm of y or z, &c.; in fact,

$$\log x + \psi(y', z' \ldots) = \log y + \log \frac{x}{y} + \psi(y', z' \ldots)$$
$$= \log y + \chi(y', z' \ldots).$$

II.—TRANSFORMATION OF THE DIFFERENTIAL EQUATIONS OF A PLANET'S MOTION.

By B. Bronwin.

Making x, y, z the co-ordinates of a disturbed planet, x', y', z' those of the disturbing body, r and r' the radii-vectores, and

$$R = \frac{m'(xx' + yy' + zz')}{r'^3} - \frac{m'}{\{r'^2 - 2(xx' + yy' + zz') + r^2\}^{\frac{1}{2}}};$$

the known differential equations of its motion are

$$\frac{d^2x}{dt^2} + \frac{\mu x}{r^3} + \frac{dR}{dx} = 0, \quad \frac{d^2y}{dt^2} + \frac{\mu y}{r^3} + \frac{dR}{dy} = 0, \quad \frac{d^2z}{dt^2} + \frac{\mu z}{r^3} + \frac{dR}{dz} = 0 \quad \ldots\ldots (A).$$

Let i be the inclination of the plane of the orbit to the fixed plane of xy, θ the longitude of the node on the fixed plane, and ϑ its longitude on the plane of the orbit, having a fixed origin on it. Then $np = nn' \cos n'np$ (see fig. 1), or $d\vartheta = \cos i \, d\theta$.

If u and v be the co-ordinates referred to the line of the nodes, $x = u\cos\theta - v\sin\theta$, $y = u\sin\theta + v\cos\theta$. And if w be the co-ordinate on the orbit, of which v is the projection, $v = w\cos i$, $z = w\sin i$. Also, if ξ and η be the co-ordinates on the plane of the orbit, the axis of η passing through the origin of ϑ,

$$u = \xi\cos\vartheta + \eta\sin\vartheta, \quad w = -\xi\sin\vartheta + \eta\cos\vartheta.$$

From these, by eliminating u, v, and w, we have

$$x = A\xi + B\eta, \quad y = C\xi + D\eta, \quad z = E\xi + F\eta \ldots\ldots (1),$$

$A = \cos\theta\cos\vartheta + \cos i\sin\theta\sin\vartheta$, $B = \cos\theta\sin\vartheta - \cos i\sin\theta\cos\vartheta$,
$C = \sin\theta\cos\vartheta - \cos i\cos\theta\sin\vartheta$, $D = \sin\theta\sin\vartheta + \cos i\cos\theta\cos\vartheta$,
$E = -\sin i\sin\vartheta$, $F = \sin i\cos\vartheta$.

Remembering that $d\vartheta = \cos i\, d\theta$, we find

$$dA = -\sin\theta\cos\theta\, d\vartheta - \cos i\cos\theta\sin\vartheta\, d\theta - \sin i\sin\theta\sin\vartheta\, di + \cos i\cos\theta\sin\vartheta\, d\theta + \cos^2 i\sin\theta\cos\vartheta\, d\theta$$

$$= -\sin i\sin\theta\sin\vartheta\, di - \sin^2 i\sin\theta\cos\vartheta\, d\theta;$$

$$\text{and } dE = -\cos i\sin\vartheta\, di - \sin i\cos i\cos\vartheta\, d\theta.$$

From these it will be easily seen that $dA = \tan i\sin\theta\, dE$.

Similar results may be obtained by differentiating the values of B, C, &c. They are, including the one just obtained,

$$\left.\begin{aligned} dA &= \tan i\sin\theta\, dE, \quad dB = \tan i\sin\theta\, dF \\ dC &= -\tan i\cos\theta\, dE, \quad dD = -\tan i\cos\theta\, dF \end{aligned}\right\} \ldots\ldots (2).$$

As we have four quantities to determine and only three equations, we must make an assumption. Assume

$$\xi dE + \eta dF = 0,\ \xi dA + \eta dB = 0,\ \xi dC + \eta dD = 0 \ldots (3).$$

This is only equivalent to one assumption: for if we put for dA, dB, dC, dD their values from (2), we see that the two last of (3) reduce to the first. The assumption here made is the one usually made. For if ϕ = lat. v = long. on the orbit, we have $z = r \sin \phi$, $\xi = r \cos v$, $\eta = r \sin v$; and $z = E\xi + F\eta$ becomes

$$\sin \phi = E \cos v + F \sin v = \sin i \,(\sin v \cos \vartheta - \cos v \sin \vartheta)$$
$$= \sin i \sin (v - \vartheta);$$

$$\text{and } \xi dE + \eta dF = 0 \text{ is } \frac{d \sin \phi}{di}\, di + \frac{d \sin \phi}{d\vartheta}\, d\vartheta = 0.$$

By virtue of (3) we have

$$\frac{dx}{dt} = A \frac{d\xi}{dt} + B \frac{d\eta}{dt},\ \frac{dy}{dt} = C \frac{d\xi}{dt} + D \frac{d\eta}{dt},\ \frac{dz}{dt} = E \frac{d\xi}{dt} + F \frac{d\eta}{dt};$$

and therefore

$$\left.\begin{aligned} \frac{d^2x}{dt^2} &= A \frac{d^2\xi}{dt^2} + B \frac{d^2\eta}{dt^2} + \frac{dA}{dt} \frac{d\xi}{dt} + \frac{dB}{dt} \frac{d\eta}{dt} \\ \frac{d^2y}{dt^2} &= C \frac{d^2\xi}{dt^2} + D \frac{d^2\eta}{dt^2} + \frac{dC}{dt} \frac{d\xi}{dt} + \frac{dD}{dt} \frac{d\eta}{dt} \\ \frac{d^2z}{dt^2} &= E \frac{d^2\xi}{dt^2} + F \frac{d^2\eta}{dt^2} + \frac{dE}{dt} \frac{d\xi}{dt} + \frac{dF}{dt} \frac{d\eta}{dt} \end{aligned}\right\} \ldots (4).$$

The following equations of condition will now be necessary:

$$\left.\begin{aligned} &A^2 + C^2 + E^2 = 1,\ B^2 + D^2 + F^2 = 1,\ AB + CD + EF = 0, \\ &AdA + CdC + EdE = 0,\quad BdB + DdD + FdF = 0, \\ &AdB + CdD + EdF = 0,\quad BdA + DdC + FdE = 0, \end{aligned}\right\} \ldots (5).$$

These may be derived as follows. In $r^2 = x^2 + y^2 + z^2 = \xi^2 + \eta^2$ put for x, y, and z their values from (1); the coefficients of ξ^2, $\xi\eta$, η^2, equalled to zero, will give the three first. Differentiation of the first and second will give the fourth and fifth: these last, by means of (3), will give the sixth and seventh. Or if in $xdx + ydy + zdz = \xi d\xi + \eta d\eta$ for x, dx, &c. we substitute their values from (1), the coefficients of the several terms of the result equalled to zero will give them all. And they may be all verified by putting for A, B, &c. and their differentials their values.

We easily derive from (4), attending to (5),

$$\left.\begin{aligned} A\frac{d^2x}{dt^2} + C\frac{d^2y}{dt^2} + E\frac{d^2z}{dt^2} &= \frac{d^2\xi}{dt^2} \\ B\frac{d^2x}{dt^2} + D\frac{d^2y}{dt^2} + F\frac{d^2z}{dt^2} &= \frac{d^2\eta}{dt^2} \end{aligned}\right\} \ldots\ldots (6),$$

and from (1), having regard to (5),

$$\frac{\mu}{r^3}(Ax + Cy + Ez) = \frac{\mu\xi}{r^3}, \quad \frac{\mu}{r^3}(Bx + Dy + Fz) = \frac{\mu\eta}{r^3} \ldots (7).$$

Multiply the first of (*A*) by *A*, the second by *C*, the third by *E*, and add the results; we have by (6) and (7),

$$\frac{d^2\xi}{dt^2} + \frac{\mu\xi}{r^3} + A\frac{dR}{dx} + C\frac{dR}{dy} + E\frac{dR}{dz} = 0.$$

Again, multiply the first of (*A*) by *B*, the second by *D*, the third by *F*; add results: we have by (6) and (7),

$$\frac{d^2\eta}{dt^2} + \frac{\mu\eta}{r^3} + B\frac{dR}{dx} + D\frac{dR}{dy} + F\frac{dR}{dz} = 0.$$

But

$$\frac{dR}{d\xi} = \frac{dR}{dx}\frac{dx}{d\xi} + \frac{dR}{dy}\frac{dy}{d\xi} + \frac{dR}{dz}\frac{dz}{d\xi} = A\frac{dR}{dx} + C\frac{dR}{dy} + E\frac{dR}{dz} \text{ by (1).}$$

In like manner $\dfrac{dR}{d\eta} = B\dfrac{dR}{dx} + D\dfrac{dR}{dy} + F\dfrac{dR}{dz}$.

These values of $\dfrac{dR}{d\xi}$, $\dfrac{dR}{d\eta}$ reduce the equations last obtained to

$$\frac{d^2\xi}{dt^2} + \frac{\mu\xi}{r^3} + \frac{dR}{d\xi} = 0, \quad \frac{d^2\eta}{dt^2} + \frac{\mu\eta}{r^3} + \frac{dR}{d\eta} = 0 \ldots (B).$$

The equations (*B*) are the same as they would be if the plane of the orbit were fixed.

Since *R* is independent of the position of the fixed plane, if we make that plane coincide with the orbit, the quantity $xx' + yy' + zz'$ will become $x'\xi + y'\eta$, or $M\xi + N\eta$. And if we transform that quantity, it will become $M\xi + N\eta$, *M* and *N* having the same values in both cases, though functions of different quantities in the two cases. Consequently $\dfrac{dR}{d\xi}$, $\dfrac{dR}{d\eta}$ will have the same values, whatever be the position of the fixed plane.

Make $\dfrac{\xi d\eta - \eta d\xi}{dt} = h$. Then by (3),

$$\frac{dA}{dt}\frac{d\xi}{dt} + \frac{dB}{dt}\frac{d\eta}{dt} = \frac{dA}{dt}\cdot\frac{\eta d\xi - \xi d\eta}{\eta dt} = -\frac{h}{\eta}\frac{dA}{dt},$$

$$\frac{dC}{dt}\frac{d\xi}{dt} + \frac{dD}{dt}\frac{d\eta}{dt} = \frac{dC}{dt}\cdot\frac{\eta d\xi - \xi d\eta}{\eta dt} = -\frac{h}{\eta}\frac{dC}{dt},$$

$$\frac{dE}{dt}\frac{d\xi}{dt} + \frac{dF}{dt}\frac{d\eta}{dt} = \frac{dE}{dt}\cdot\frac{\eta d\xi - \xi d\eta}{\eta dt} = -\frac{h}{\eta}\frac{dE}{dt}.$$

Substituting these values in (4), they become

$$\left.\begin{aligned}\frac{d^2x}{dt^2} &= A\frac{d^2\xi}{dt^2} + B\frac{d^2\eta}{dt^2} - \frac{h}{\eta}\frac{dA}{dt}\\ \frac{d^2y}{dt^2} &= C\frac{d^2\xi}{dt^2} + D\frac{d^2\eta}{dt^2} - \frac{h}{\eta}\frac{dC}{dt}\\ \frac{d^2z}{dt^2} &= E\frac{d^2\xi}{dt^2} + F\frac{d^2\eta}{dt^2} - \frac{h}{\eta}\frac{dE}{dt}\end{aligned}\right\} \quad \ldots\ (8).$$

These last may be changed by (3) into

$$\left.\begin{aligned}\frac{d^2x}{dt^2} &= A\frac{d^2\xi}{dt^2} + B\frac{d^2\eta}{dt^2} + \frac{h}{\xi}\frac{dB}{dt}\\ \frac{d^2y}{dt^2} &= C\frac{d^2\xi}{dt^2} + D\frac{d^2\eta}{dt^2} + \frac{h}{\xi}\frac{dD}{dt}\\ \frac{d^2z}{dt^2} &= E\frac{d^2\xi}{dt^2} + F\frac{d^2\eta}{dt^2} + \frac{h}{\xi}\frac{dF}{dt}\end{aligned}\right\} \quad \ldots\ (9).$$

By (2) we find

$$\left.\begin{aligned}\left(\frac{dA}{dt}\right)^2 + \left(\frac{dC}{dt}\right)^2 + \left(\frac{dE}{dt}\right)^2 &= \frac{1}{\cos^2 i}\left(\frac{dE}{dt}\right)^2\\ \left(\frac{dB}{dt}\right)^2 + \left(\frac{dD}{dt}\right)^2 + \left(\frac{dF}{dt}\right)^2 &= \frac{1}{\cos^2 i}\left(\frac{dF}{dt}\right)^2\end{aligned}\right\} \quad \ldots\ (10).$$

We now easily find from (8) and (9), having regard to (5) and (10),

$$\left.\begin{aligned}\frac{dA}{dt}\frac{d^2x}{dt^2} + \frac{dC}{dt}\frac{d^2y}{dt^2} + \frac{dE}{dt}\frac{d^2z}{dt^2} &= -\frac{h}{\eta\cos^2 i}\left(\frac{dE}{dt}\right)^2\\ \frac{dB}{dt}\frac{d^2x}{dt^2} + \frac{dD}{dt}\frac{d^2y}{dt^2} + \frac{dF}{dt}\frac{d^2z}{dt^2} &= \frac{h}{\xi\cos^2 i}\left(\frac{dF}{dt}\right)^2\end{aligned}\right\} \quad \ldots\ (11).$$

Putting for x, y, and z their values from (1), we find in consequence of (5),

$$\left.\begin{aligned} \frac{\mu}{r^3}\left(\frac{dA}{dt}x+\frac{dC}{dt}y+\frac{dE}{dt}z\right)&=0\\ \frac{\mu}{r^3}\left(\frac{dB}{dt}x+\frac{dD}{dt}y+\frac{dF}{dt}z\right)&=0 \end{aligned}\right\} \quad \ldots\ldots (12).$$

Multiply first of (*A*) by $\frac{dA}{dt}$, second by $\frac{dC}{dt}$, third by $\frac{dE}{dt}$; add results: by (11) and (12) we obtain

$$-\frac{h}{\eta\cos^2 i}\left(\frac{dE}{dt}\right)^2+\frac{dA}{dt}\frac{dR}{dx}+\frac{dC}{dt}\frac{dR}{dy}+\frac{dE}{dt}\frac{dR}{dz}.$$

Again, multiply first of (*A*) by $\frac{dB}{dt}$, second by $\frac{dD}{dt}$, third by $\frac{dF}{dt}$; add results: by (11) and (12) we obtain

$$\frac{h}{\xi\cos^2 i}\left(\frac{dF}{dt}\right)^2+\frac{dB}{dt}\frac{dR}{dx}+\frac{dD}{dt}\frac{dR}{dy}+\frac{dF}{dt}\frac{dR}{dz}.$$

Putting for dA, dB, dC, dD their values from (2), the two last found equations reduce to

$$\left.\begin{aligned} \frac{h}{\cos i}\frac{dE}{dt}&=\eta\sin i\sin\theta\frac{dR}{dx}-\eta\sin i\cos\theta\frac{dR}{dy}+\eta\cos i\frac{dR}{dz}\\ \frac{h}{\cos i}\frac{dF}{dt}&=-\xi\sin i\sin\theta\frac{dR}{dx}+\xi\sin i\cos\theta\frac{dR}{dy}-\xi\cos i\frac{dR}{dz} \end{aligned}\right\}$$

Now, by (1) and (2), $\ldots\ldots (13),$

$$\left.\begin{aligned} \frac{dR}{dF}=\frac{dR}{dx}\frac{dx}{dF}+\frac{dR}{dy}\frac{dy}{dF}+\frac{dR}{dz}\frac{dz}{dF}&=\eta\frac{dR}{dx}\frac{dB}{dF}+\eta\frac{dR}{dy}\frac{dD}{dF}+\eta\frac{dR}{dz}\\ &=\eta\tan i\sin\theta\frac{dR}{dx}-\eta\tan i\cos\theta\frac{dR}{dy}+\eta\frac{dR}{dz}. \end{aligned}\right\}$$

In like manner

$$\frac{dR}{dE}=\xi\tan i\sin\theta\frac{dR}{dx}-\xi\tan i\cos\theta\frac{dR}{dy}+\xi\frac{dR}{dz}.$$

By means of (14), (13) will become $\ldots\ldots (14).$

$$\frac{h}{\cos^2 i}\frac{dE}{dt}=\frac{dR}{dF},\quad \frac{h}{\cos^2 i}\frac{dF}{dt}=-\frac{dR}{dE}\ldots\ldots (15).$$

To follow a usual notation, make

$$p=-E=\sin i\sin\vartheta,\quad q=F=\sin i\cos\vartheta,$$

and (15) become

$$dp = -\frac{\cos^2 i\, dt}{h}\frac{dR}{dq}, \quad dq = \frac{\cos^2 i\, dt}{h}\frac{dR}{dp} \ldots . (16),$$

where $\cos^2 i = 1 - p^2 - q^2$.

From equations (16) the following formulæ, which are sometimes used, may easily be deduced:

$$d\theta = -\frac{dt}{h \sin i}\frac{dR}{di}, \quad d\Im = -\frac{\cos i\, dt}{h \sin i}\frac{dR}{di}, \quad di = \frac{dt}{h \sin i}\frac{dR}{d\theta}.$$

The equation $\sin \phi = E \cos v + F \sin v = \sin i \sin (v - \Im)$ will give the latitude by means of E and F, and we may form a differential equation for the purpose if we please. In which case we may find the variations of E and F, or of i and $\Im$, due to the several powers of the disturbing force, from those of ϕ and v, in the manner pointed out in my paper on the Differential Equations of the Moon's motion in the *London, Edinburgh, and Dublin Philosophical Magazine*, Feb. 1844. Those equations are founded on the previous knowledge of equations (B) of this paper; which last are convenient for finding the co-ordinates r and v on the plane of the orbit instead of their projected values on the fixed plane. And this method is mostly the easiest way of solving the problem.

III.—ON A NEW METHOD OF OBTAINING THE EXPRESSION FOR THE SINE AND COSINE OF THE MULTIPLE ARC IN TERMS OF THE LIKE FUNCTIONS OF THE SIMPLE ARC.

By R. Moon, M.A. Fellow of Queens' College.

The subject which I propose to discuss in the following short paper is somewhat trite; but as the method I purpose to adopt possesses great simplicity, and as the artifice adopted in it may be usefully extended to other cases, I perhaps may be excused in my endeavour to call attention to it.

By De Moivre's Theorem,

$$\cos n\theta + \sqrt{(-1)} \sin n\theta = \{\cos \theta + \sqrt{(-1)} \sin \theta\}^n$$
$$= \cos^n \theta + C_1 \sqrt{(-1)} \sin \theta \cos^{n-1} \theta - C_2 \sin^2 \theta \cos^{n-2} \theta$$
$$- C_3 \sqrt{(-1)} \sin^3 \theta \cos^{n-3} \theta + \&c.$$

where C_1, C_2, C_3 .... are the successive coefficients in the expansion of $(1 + x)^n$;

Hence $\cos n\theta = \cos^n\theta - C_2 \sin^2\theta \cos^{n-2}\theta + C_4 \sin^4\theta \cos^{n-4}\theta$ – &c.

It will be observed, that the coefficient of $\cos^{n-2r}\theta$ in the above expression is a homogeneous function of one dimension of the quantities C_2, C_3, C_4 . . . . ; and this function will be identical with the coefficient of $\cos^{n-2r}\theta$ in the expression

$$u = \cos^n\theta - A(1-\cos^2\theta)\cos^{n-2}\theta + A^2(1-\cos^2\theta)^2\cos^{n-4}\theta - \&c.$$

if we change A into C_2, A^2 into C_4, A^3 into C_6, &c. But

$$u = \cos^n\theta \left\{1 - A\frac{1-\cos^2\theta}{\cos^2\theta} + A^2\left(\frac{1-\cos^2\theta}{\cos^2\theta}\right)^2 - A^3\left(\frac{1-\cos^2\theta}{\cos^2\theta}\right)^3 + \&c.\right\}$$

$$= \cos^n\theta \left\{f\left(-\left(A\frac{1-\cos^2\theta}{\cos^2\theta}\right)\right)\right\}$$

$$= \cos^n\theta \left\{fA - \frac{A}{\cos^2\theta}\right\}$$

$$= \cos^n\theta \left\{fA - \frac{1}{1}\frac{A}{\cos^2\theta}\frac{dfA}{dA} + \frac{1}{1.2}\left(\frac{A}{\cos^2\theta}\right)^2\frac{d^2fA}{dA^2} - \frac{1}{1.2.3}\left(\frac{A}{\cos^2\theta}\right)^3\frac{d^3fA}{dA^3} + \&c.\right\}$$

and fA is the coefficient of $\cos^n\theta$ in the equivalent for u; therefore

$$fA = 1 + A + A^2 + A^3 + A^4 + \&c.$$

$$\frac{dfA}{dA} = 1 + 2A + 3A^2 + 4A^3 + \&c.$$

$$\frac{d^2fA}{dA^2} = 2.1 + 3.2A + 4.3A^2 + \&c.$$

$$\frac{d^3fA}{d\theta^3} = 3.2.1 + 4.3.2A + \&c.$$

$$\ldots = \ldots$$

Hence

$$u = \cos^n\theta(1 + A + A^2 + A^3 + \&c.) - \frac{\cos^{n-2}\theta}{1}(A + 2A^2 + 3A^3 + 4A^4 + \&c.) + \frac{\cos^{n-4}\theta}{1.2}(1.2A^2 + 2.3A^3 + 3.4A^4 + \&c.) - \frac{\cos^{n-6}\theta}{1.2.3}(1.2.3A^3 + 2.3.4A^4 + 3.4.5A^5 + \&c.) + \&c.;$$

and substituting for A, A^2, A^3 .... &c., we get

$$\begin{aligned}\cos n\theta = {} & \cos^n\theta\,(1 + C_2 + C_4 + C_6 + \&c.)\\ & - \frac{\cos^{n-2}\theta}{1}(C_2 + 2C_4 + 3C_6 + \&c.)\\ & + \frac{\cos^{n-4}\theta}{1.2}(1.2C_4 + 2.3C_6 + 3.4C_8 + \&c.)\\ & - \frac{\cos^{n-6}\theta}{1.2.3}(1.2.3C_6 + 2.3.4C_8 + 3.4.5C_{10} + \&c.\\ & + \&c.\end{aligned}$$

a result which may be readily verified.

Similarly we have

$$\begin{aligned}\sin n\theta &= C_1\sin\theta\cos^{n-1}\theta - C_3\sin^3\theta\cos^{n-3}\theta + C_5\sin^5\theta\cos^{n-5}\theta - \&c.\\ &= \sin\theta\,\{C_1\cos^{n-1}\theta - C_3(1-\cos^2\theta)\cos^{n-3}\theta\\ &\qquad + C_5(1-\cos^2\theta)^2\cos^{n-5}\theta - \&c.\\ &= \sin\theta\,.\,u.\end{aligned}$$

Now the coefficient of $(\cos\theta)^{n-p}$ in u may be found from that of $(\cos\theta)^{n-p}$ in the expression

$$v = A\cos^{n-1}\theta - A^2(1-\cos^2\theta)\cos^{n-3}\theta + A^3(1-\cos^2\theta)^2\cos^{n-5}\theta - \&c.$$

by substituting C_2 for A, C_3 for A^2, C_5 for A^3, &c.

But $$v = A\cos^{n-1}\theta\left\{1 - A\frac{1-\cos^2\theta}{\cos^2\theta} + A^2\left(\frac{1-\cos^2\theta}{\cos^2\theta}\right)^2 - \&c.\right\}$$

$$= A\cos^{n-1}\theta\, f\left(A - \frac{A}{\cos^2\theta}\right);$$

whence we easily find

$$\begin{aligned}\sin n\theta = \sin\theta\,\Big\{ & \cos^{n-1}\theta\,(C_1 + C_3 + C_5 + \&c.)\\ & - \frac{\cos^{n-3}\theta}{1}(C_3 + 2C_5 + 3C_7 + \&c.)\\ & + \frac{\cos^{n-5}\theta}{1.2}(1.2C_5 + 2.3C_7 + 3.4C_9 + \&c.)\\ & - \frac{\cos^{n-7}\theta}{1.2.3}(1.2.3C_7 + 2.3.4C_9 + \&c.) + \&c.\end{aligned}$$

We might in the same manner easily obtain the expression for $\sin n\theta$ and $\cos n\theta$ in terms of the powers of $\sin\theta$.

Liverpool, Dec. 20, 1844.

IV.—ON INDETERMINATE MAXIMA AND MINIMA.

By W. Walton, M.A., Trinity College.

Suppose that we have n equations connecting a quantity r with n quantities $x, y, z, \ldots$ and let it be proposed to determine a relation among certain parameters of these equations, such that for all values of $x, y, z, \ldots$ the value of r shall remain invariable. Problems coming under this general head may generally be solved with much elegance by the following method. Proceed by the ordinary rules to find the values of $x, y, z, \ldots$ which correspond to a maximum or minimum value of r: the result of the investigation will be an equation which, together with the n original equations, will generally serve to determine $x, y, z, \ldots$ as well as r. Having obtained these equations, we must assign such relations among the proper parameters of the equations as shall render the values of $x, y, z, \ldots$ all or some of them, according to the case, indeterminate. These relations will constitute the solution of the problem proposed. I will subjoin the solutions of several problems of the above class, which will sufficiently elucidate the principles of the method.

(1) To investigate the positions of the circular sections of the surface $(x^2 + y^2 + z^2)^2 = a^2x^2 + b^2y^2 + c^2z^2 \ldots\ldots\ldots (1)$,

made by planes passing through the centre of the surface.

Let (l, m, n) be the direction-cosines of any plane section through the origin, r any radius of the section, and α, β, γ, its direction-cosines. Then

$$l\alpha + m\beta + n\gamma = 0 \ldots\ldots\ldots\ldots\ldots (2),$$

and

$$\alpha^2 + \beta^2 + \gamma^2 = 1 \ldots\ldots\ldots\ldots\ldots (3);$$

also, from (1), we have

$$r^2 = a^2\alpha^2 + b^2\beta^2 + c^2\gamma^2 \ldots\ldots\ldots\ldots (4).$$

Since r must be constant for a circular section, differentiating (2), (3), (4), on this hypothesis with respect to α, β, γ, we have

$$l d\alpha + m d\beta + n d\gamma = 0,$$

$$\alpha d\alpha + \beta d\beta + \gamma d\gamma = 0,$$

$$a^2\alpha d\alpha + b^2\beta d\beta + c^2\gamma d\gamma = 0.$$

Multiplying these three equations in order by $1, -\lambda\mu, -\lambda$, and adding, we get, $\lambda\mu$ and λ being arbitrary multipliers,

$$\left.\begin{aligned} l &= \lambda(\mu - a^2)\alpha \\ m &= \lambda(\mu - b^2)\beta \\ n &= \lambda(\mu - c^2)\gamma \end{aligned}\right\} \ldots\ldots\ldots\ldots (5).$$

From (2) and (5) there is

$$\frac{l^2}{\mu - a^2} + \frac{m^2}{\mu - b^2} + \frac{n^2}{\mu - c^2} = 0 \ldots\ldots\ldots (6):$$

from (3) and (5),

$$\frac{l^2}{(\mu - a^2)^2} + \frac{m^2}{(\mu - b^2)^2} + \frac{n^2}{(\mu - c^2)^2} = \lambda^2 \ldots\ldots (7).$$

From (3) and (4),

$$\lambda^2 r^2 = \frac{l^2 a^2}{(\mu - a^2)^2} + \frac{m^2 b^2}{(\mu - b^2)^2} + \frac{n^2 c^2}{(\mu - c^2)^2}$$

$$= \mu\lambda^2, \quad \text{by } \mu(7) - (6),$$

whence

$$r^2 = \mu \ldots\ldots\ldots\ldots\ldots\ldots (8).$$

Now, the section being circular, α, β, γ, must each be an indeterminate quantity: hence, as is evident from (5), the value of λ in (7) must be of the form $\frac{0}{0}$. We must have, therefore, one of the three relations

$$\left.\begin{matrix} l^2 = 0 \\ r^2 = a^2 \end{matrix}\right\}, \quad \left.\begin{matrix} m^2 = 0 \\ r^2 = b^2 \end{matrix}\right\}, \quad \left.\begin{matrix} n^2 = 0 \\ r^2 = c^2 \end{matrix}\right\}.$$

Corresponding to each of these relations, we must have respectively, from (2), (5), (8),

$$\frac{m^2}{a^2 - b^2} + \frac{n^2}{a^2 - c^2} = 0, \quad \frac{n^2}{b^2 - c^2} + \frac{l^2}{b^2 - a^2} = 0, \quad \frac{l^2}{c^2 - a^2} + \frac{m^2}{c^2 - b^2} = 0:$$

of which relations the second alone is possible. Thus we see that there are two circular sections given by the equation

$$\frac{l}{n} = \pm \left(\frac{a^2 - c^2}{b^2 - c^2}\right)^{\frac{1}{2}}, \quad m = 0;$$

the radii of both being equal to b.

If any other possible relation were established between l, m, n, the values of α, β, γ, r, might be found from the equations (5), (6), (7), (8); so that r would be ascertained both in magnitude and position. Thus the problem with which we have been occupied is a nugatory case of a general problem of maximum or minimum values.

(2) To investigate the positions of the circular sections of an ellipsoid made by planes passing through the centre.

The equation to the ellipsoid being

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

whence $$\frac{1}{r^2} = \frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} + \frac{\gamma^2}{c^2},$$

it is evident that all the equations of the last problem will be adapted to the case of the ellipsoid by writing $\frac{1}{r^2}$, $\frac{1}{a^2}$, $\frac{1}{b^2}$, $\frac{1}{c^2}$, respectively in place of r^2, a^2, b^2, c^2.

Thus we see that the positions of the circular sections will be determined by one of the relations

$$\frac{m^2}{\frac{1}{a^2}-\frac{1}{b^2}} + \frac{n^2}{\frac{1}{a^2}-\frac{1}{c^2}} = 0, \quad \frac{n^2}{\frac{1}{b^2}-\frac{1}{c^2}} + \frac{l^2}{\frac{1}{b^2}-\frac{1}{a^2}} = 0, \quad \frac{l^2}{\frac{1}{c^2}-\frac{1}{a^2}} + \frac{m^2}{\frac{1}{c^2}-\frac{1}{b^2}} = 0,$$

the second only of which is possible. Thus we see that two circular sections are determined by the equations

$$m = 0, \quad \frac{l}{n} = \pm \left\{ \frac{\frac{1}{b^2}-\frac{1}{a^2}}{\frac{1}{c^2}-\frac{1}{b^2}} \right\}^{\frac{1}{2}}.$$

(3) To find the position of a plane passing through the centre of an ellipsoid, such that the perpendiculars from the centre upon the tangent planes at every point of the curve of section shall be equal.

The equation to the ellipsoid being

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \ldots\ldots\ldots (1),$$

we shall have, p being one of the perpendiculars,

$$\frac{1}{p^2} = \frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4} \ldots\ldots\ldots (2).$$

Let the equation to the plane of section be

$$lx + my + nz = 0 \ldots\ldots (3).$$

Since p is to be constant, we have from (3), (1), (2),

$$l\,dx + m\,dy + n\,dz = 0,$$

$$\frac{x}{a^2}\,dx + \frac{y}{b^2}\,dy + \frac{z}{c^2}\,dz = 0,$$

$$\frac{x}{a^4}\,dx + \frac{y}{b^4}\,dy + \frac{z}{c^4}\,dz = 0.$$

Multiplying these equations by 1, $-\lambda\mu$, $-\lambda$, and adding, we shall get, $\lambda\mu$ and λ being arbitrary,

$$\left.\begin{aligned} l &= \frac{\lambda}{a^2}\left(\mu - \frac{1}{a^2}\right)x, \\ m &= \frac{\lambda}{b^2}\left(\mu - \frac{1}{b^2}\right)y, \\ n &= \frac{\lambda}{c^2}\left(\mu - \frac{1}{c^2}\right)z. \end{aligned}\right\} \quad \ldots\ldots (4).$$

From (1) and (4),

$$\frac{l^2a^2}{\left(\mu - \frac{1}{a^2}\right)^2} + \frac{m^2b^2}{\left(\mu - \frac{1}{b^2}\right)^2} + \frac{n^2c^2}{\left(\mu - \frac{1}{c^2}\right)^2} = \lambda^2 \ldots\ldots (5);$$

from (3) and (4),

$$\frac{l^2a^2}{\mu - \frac{1}{a^2}} + \frac{m^2b^2}{\mu - \frac{1}{b^2}} + \frac{n^2c^2}{\mu - \frac{1}{c^2}} = 0 \quad \ldots\ldots\ldots (6).$$

From (2) and (4),

$$\frac{\lambda^2}{p^2} = \frac{l^2}{\left(\mu - \frac{1}{a^2}\right)^2} + \frac{m^2}{\left(\mu - \frac{1}{b^2}\right)^2} + \frac{n^2}{\left(\mu - \frac{1}{c^2}\right)^2}$$

$$= \mu\lambda^2, \text{ by } \mu\,(5) - (6),$$

whence

$$\frac{1}{p^2} = \mu \ldots\ldots\ldots\ldots\ldots (7).$$

Since λ, given by (5), must be indeterminate, in order to render x, y, z, indeterminate in (4), we must have one of the relations

$$\left\{\begin{aligned} l &= 0 \\ \mu &= \frac{1}{a^2} \end{aligned}\right\}, \quad \left\{\begin{aligned} m &= 0 \\ \mu &= \frac{1}{b^2} \end{aligned}\right\}, \quad \left\{\begin{aligned} n &= 0 \\ \mu &= \frac{1}{c^2} \end{aligned}\right\}.$$

Corresponding respectively to these relations we have, from (3), (4), (7),

$$\frac{m^2b^2}{\frac{1}{a^2} - \frac{1}{b^2}} + \frac{n^2c^2}{\frac{1}{a^2} - \frac{1}{c^2}} = 0, \quad \frac{n^2c^2}{\frac{1}{b^2} - \frac{1}{c^2}} + \frac{l^2a^2}{\frac{1}{b^2} - \frac{1}{a^2}} = 0,$$

$$\frac{l^2a^2}{\frac{1}{c^2} - \frac{1}{a^2}} + \frac{m^2b^2}{\frac{1}{c^2} - \frac{1}{b^2}} = 0,$$

of which the second alone is possible.

Thus we see that there are two sections determined by the equations

$$m = 0, \quad \frac{l}{n} = \pm \frac{c}{a} \left\{ \frac{\frac{1}{b^2} - \frac{1}{a^2}}{\frac{1}{c^2} - \frac{1}{b^2}} \right\}^{\frac{1}{2}},$$

the constant magnitude of the perpendicular being $= b^2$.

V.—ON THE INVERSE ELLIPTIC FUNCTIONS.

By A. Cayley, M.A. Fellow of Trinity College.

The properties of the inverse elliptic functions have been the object of the researches of the two illustrious analysts, Abel and Jacobi. Among their most remarkable ones may be reckoned the formulæ given by Abel (*Œuvres*, t. I. p. 213), in which the functions ϕa, fa, Fa, (corresponding to Jacobi's sin $am.a$, cos $am.a$, $\Delta am.a$, though not precisely equivalent to these, Abel's radical being $[(1 - c^2x^2)(1 + e^2x^2)]^{\frac{1}{2}}$, and Jacobi's, like that of Legendre's $[(1 - x^2)(1 - k^2x^2)]^{\frac{1}{2}}$), are expressed in the form of fractions, having a common denominator; and this, together with the three numerators, resolved into a doubly infinite series of factors; *i.e.* the general factor contains two independent integers. These formulæ may conveniently be referred to as "Abel's double factorial expressions" for the functions ϕ, f, F. By dividing each of these products into an infinite number of partial products, and expressing these by means of circular or exponential functions, Abel has obtained (p. 216–218) two other systems of formulæ for the same quantities, which may be referred to as "Abel's first and second single factorial systems." The theory of the functions forming the above numerators and denominator, is mentioned by Abel in a letter to Legendre (*Œuvres*, t. II. p. 259), as a subject to which his attention had been directed, but none of his researches upon them have ever been published. Abel's double factorial expressions have nowhere any thing analogous to them in Jacobi's *Fund. Nova;* but the system of formulæ analogous to the first single factorial system is given by Jacobi (p. 86), and the second system is implicitly contained in some of the subsequent formulæ. The functions forming the numerator and denominator of sin $am.u$, Jacobi represents, omitting a constant factor, by $H,(u)$, $\Theta(u)$; and proceeds to investigate the properties of these new functions. This he principally effects by means of a very remarkable equation of the form

$$l\Theta(u) = \tfrac{1}{2}Au^2 + B\int_0 du . \int_0 du \sin^2 amu,$$

(*Fund. Nova*, p. 145, 133), by which $\Theta(u)$ is made to depend on the known function sin *am.u*. The other two numerators are easily expressed by means of the two functions H, Θ.

From the omission of Abel's double factorial expressions, which are the only ones which display clearly the real nature of the functions in the numerators and denominators; and besides, from the different form of Jacobi's radical, which complicates the transformation from an impossible to a possible argument, it is difficult to trace the connection between Jacobi's formulæ; and in particular to account for the appearance of an exponential factor which runs through them. It would seem therefore natural to make the whole theory depend upon the definitions of the new transcendental functions to which Abel's double factorial expressions lead one, even if these definitions were not of such a nature, that one only wonders they should never have been assumed *à priori* from the analogy of the circular functions sin. cos. and quite independently of the theory of elliptic integrals. This is accordingly what I have done in the present paper, in which therefore I assume no single property of elliptic functions, but demonstrate them all, from my fundamental equations. For the sake however of comparison, I retain entirely the notation of Abel. Several of the formulæ that will be obtained are new.

The infinite product

$$x\Pi\left(1+\frac{x}{m\omega}\right)\ldots\ldots\ldots\ldots\ldots(1),$$

where m receives the integer values $\pm 1, \pm 2, \ldots \pm r$, converges, as is well known, as r becomes indefinitely great to a determinate function $\sin\frac{\pi x}{\omega}$ of x; the theory of which might, if necessary, be investigated from this property assumed as a definition. We are thus naturally led to investigate the properties of the new transcendant

$$u = x\Pi\Pi\left(1+\frac{x}{m\omega+nvi}\right)\ldots\ldots\ldots\ldots(2):$$

m and n are integer numbers, positive or negative; and it is supposed that whatever positive value is attributed to either of these, the corresponding negative one is also given to it. $i = \sqrt{(-1)}$, ω and v are real positive quantities. (At least this is the standard case, and the only one we shall explicitly consider. Many of the formulæ obtained are true, with slight modifications, whatever ω and v represent, provided only $\omega : vi$

be not a real quantity; for if it were so, $m\omega + n\nu i$ for some values of m, n would vanish, or at least become indefinitely small, and u would cease to be a determinate function of x).*

Now the value of the above expression, or, as for the sake of shortness it may be written, of the function

$$u = x\Pi\Pi\left\{1 + \frac{x}{(m, n)}\right\} \ldots\ldots\ldots\ldots (3),$$

depends in a remarkable manner on the mode in which the superior limits of m, n are assigned. Imagine m, n to have any positive or negative integer values satisfying the equation

$$\phi(m^2, n^2) < T.$$

Consider, for greater distinctness, m, n as the co-ordinates of a point; the equation $\phi(m^2, n^2) = T$ belongs to a certain curve symmetrical with respect to the two axes. I suppose besides that this is a continuous curve without multiple points, and such that the minimum value of a radius vector through the origin continually increases as T increases, and becomes infinite with T. The curve may be *analytically* discontinuous, this is of no importance. The condition with respect to the limits is then that m and n must be integer values denoting the co-ordinates of a point *within* the above curve, the whole system of such integer values being successively taken for these quantities.

Suppose, next, u' denotes the same function as u, except that the limiting condition is

$$\phi'(m^2, n^2) < T' \ldots\ldots\ldots\ldots (5).$$

The curve $\phi'(m^2, n^2) = T'$ is supposed to possess the same properties with the other limiting curve, and, for greater distinctness, to lie entirely outside of it; but this last condition is nonessential.

These conditions being satisfied, the ratio $u' : u$ is very easily determined in the limiting case of T and T' infinite. In fact

$$\frac{u'}{u} = \Pi\Pi\left\{1 + \frac{x}{(m, n)}\right\} \ldots\ldots\ldots (6),$$

or

$$\frac{lu'}{u} = \Sigma\Sigma l\left\{1 + \frac{x}{(m, n)}\right\} \ldots\ldots\ldots (7),$$

the limiting conditions being

$$\phi(m^2, n^2) > T \ldots\ldots\ldots\ldots (8).$$
$$\phi'(m^2, n^2) < T'.$$

* I have examined the case of impossible values of ω and ν in a paper which I am preparing for *Crelles' Journal*.

Now $$l\left\{1+\frac{x}{(m,n)}\right\}=\frac{x}{(m,n)}-\tfrac{1}{2}\cdot\frac{x^2}{(m,n)^2}+\ldots\ldots(9),$$

$$\frac{lu'}{u}=x.\Sigma\Sigma\frac{1}{(m,n)}-\tfrac{1}{2}x^2.\Sigma\Sigma\frac{1}{(m,n)^2}+\ldots.(10).$$

Or, the alternate terms vanishing on account of the positive and negative values destroying each other,

$$l\frac{u'}{u}=-\tfrac{1}{2}x^2.\Sigma\Sigma\frac{1}{(m,n)^2}-\tfrac{1}{4}x^4.\Sigma\Sigma\frac{1}{(m,n)^4}-\ldots..(11).$$

In general $$\Sigma\Sigma\psi(m,n)=\iint\psi(m,n)\,dmdn+P\ldots..(12),$$

P denoting a series the first term of which is of the form $C\psi(m,n)$, and the remaining ones depending on the differential coefficients of this quantity with respect to m and n. The limits between which the two sides are to be taken, are identical.

In the present case, supposing T and T' indefinitely great, it is easy to see that the first term of the expression for $l\frac{u'}{u}$ is the only one which is not indefinitely small; and we have

$$l\frac{u'}{u}=-\tfrac{1}{2}Ax^2,\quad\text{or}\quad u'=u\varepsilon^{-\frac{1}{2}Ax^2}.\ldots\ldots..(13),$$

where $$A=\iint\frac{dm.dn}{(m,n)^2}=\iint\frac{dm.dn}{(m\omega+nv i)^2}\ldots\ldots..(14);$$

the limits of the integration being given by

$$\phi(m^2,n^2)>T.\ldots\ldots\ldots\ldots..(15).$$
$$\phi'(m^2,n^2)<T'.$$

Some particular cases are important. Suppose the limits of u' are given by $m^2\omega^2<T^2,\quad n^2v^2<T^2\ldots\ldots\ldots..(16).$

And thus of u, by $m^2\omega^2+n^2v^2.<T^2\ldots\ldots\ldots\ldots..(17),$

we have $$A=\iint\frac{dmdn}{(m\omega+nv i)^2}\ldots\ldots\ldots..(18),$$

$$=-\frac{1}{\omega}\int dn.\left\{\frac{1}{T+nvi}-\frac{1}{\sqrt{(T^2-n^2v^2)+nvi}}-\frac{1}{-\sqrt{(T^2-n^2v^2)+nvi}}-\frac{1}{-T+nvi}\right\}$$

$$=-\frac{2}{\omega}\int dn.\left\{\frac{T}{T^2+n^2v^2}-\frac{\sqrt{(T^2-n^2v^2)}}{T^2}\right\}\quad(nv=-T,\ nv=T)$$

$$=-\frac{2}{\omega v}\int_{-1}^{1}d\theta.\left\{\frac{1}{1+\theta^2}-\sqrt{(1-\theta^2.)}\right\}=-\frac{2}{\omega v}(\pi-\pi)=0.$$

Or, in this case, $u' = u$ ..... (19).

Again, let the limits of u' be

$$m^2\omega^2 < R'^2, \quad n^2v^2 < S'^2 \text{ .....} (20),$$

and those of u, $m^2\omega^2 < R^2, \quad n^2v^2 < S^2$ ..... (21).

$$A = \iint \frac{dmdn}{(m\omega + nvi)^2} \text{ .....} (22)$$

$$= -\frac{1}{\omega}\int dn \cdot \left\{\frac{1}{R' + nvi} - \frac{1}{R + nvi} + \frac{1}{-R + nvi} - \frac{1}{-R' + nvi}\right\},$$

where the limits are $n^2v^2 < S'^2$, for the terms containing R', $n^2v^2 < S^2$. for the terms containing R

$$= -\frac{2}{\omega v i} l \frac{R' + S'i}{R' - S'i} \frac{R - Si}{R + Si} \text{ .....} (23),$$

$$= -\frac{4}{\omega v}(\lambda' - \lambda), \quad \text{if} \quad \lambda' = \tan^{-1}\frac{S'}{R'}, \ \lambda = \tan^{-1}\frac{S}{R},$$

the arcs λ, λ' being included between the limits $0, \frac{\pi}{2}$. Hence

$$u' = \varepsilon^{2(\lambda' - \lambda)\frac{x^2}{\omega v}} u \text{ .....} (24).$$

In particular if $\frac{S'}{R'} = \frac{S}{R}$, $u' = u$. If $\frac{S'}{R'} = 0$, $\frac{S}{R} = 1$, $u' = u\varepsilon^{-\frac{1}{2}\beta x^2}$ if $\frac{S'}{R'} = \infty$, $\frac{S}{R} = 1$, $u' = u\varepsilon^{\frac{1}{2}\beta x^2}$ where $\beta = \frac{\pi}{\omega v}$, for which quantity it will continue to be used.

We may now completely define the functions whose properties are to be investigated. Writing, for shortness,

$$(m, n) = m\omega + nvi \text{ .....} (A).$$
$$(\overline{m}, n) = (m + \tfrac{1}{2})\omega + nvi.$$
$$(m, \overline{n}) = m\omega + (n + \tfrac{1}{2})vi.$$
$$(\overline{m}, \overline{n}) = (m + \tfrac{1}{2})\omega + (n + \tfrac{1}{2})vi.$$

We may put $\gamma x = x\Pi\Pi\left\{1 + \frac{x}{(m, n)}\right\}$ ..... (B).

$$gx = \Pi\Pi\left\{1 + \frac{x}{(\overline{m}, n)}\right\}.$$

$$Gx = \Pi\Pi\left\{1 + \frac{x}{(m, \overline{n})}\right\}.$$

$$\mathfrak{G}x = \Pi\Pi\left\{1 + \frac{x}{(\overline{m}, \overline{n})}\right\}.$$

The limits being given respectively by the equations
$\mathrm{mod}^s(m,n) < T$, $\mathrm{mod}^s(\overline{m},n) < T$, $\mathrm{mod}^s(m,\overline{n}) < T$, $\mathrm{mod}^s(\overline{m},\overline{n}) < T$,
T being finally infinite. The system of values $m = 0$, $n = 0$, is of course omitted in γx.

The functions γx, gx, Gx, $\mathfrak{G}x$, are all of them real finite functions of x, possessing properties analogous to that of (u). Thus, representing any one of them by Jx, we have

$$Jx = \epsilon^{\frac{1}{2}\beta x^2} J_{\frac{1}{2}\beta} x \quad \ldots\ldots\ldots\ldots\ldots (C),$$

where $J_{\frac{1}{2}\beta}x$ is the same as Jx, only for $J_{\frac{1}{2}\beta}x$ the limits are given by $m^2\omega^2$ or $(m+\frac{1}{2})^2\omega^2 < R^2$, $n^2\upsilon^2$ or $(n+\frac{1}{2})^2\upsilon^2 < S$, ($R$, S, and $\frac{R}{S}$ infinite), and for $J_{\beta}x$, by the same formulæ, (R, S, and $\frac{S}{R}$ infinite). It is to this equation that the most characteristic properties of the functions Jx are due.

The following equations are deduced immediately from the above definitions:

$$\gamma(-x) = -\gamma x,\ g(-x) = gx,\ G(-x) = Gx,\ \mathfrak{G}(-x) = \mathfrak{G}x \ldots (D),$$

$$\gamma(0) = 0,\ g(0) = 1,\ G(0) = 1,\ \mathfrak{G}0 = 1 \quad \ldots\ldots\ldots\ldots\ldots (E),$$

$$\gamma' 0 = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (F).$$

Suppose $\gamma_1 x$, $g_1 x$, $G_1 x$, $\mathfrak{G}_1 x$, are the values that would have been obtained for γx, gx, Gx, $\mathfrak{G}x$; by interchanging ω and υ,—then changing x into xi, and interchanging m and n, by which means the limiting equations are the same in the two cases, we obtain the following system of equations:

$$\gamma_1(xi) = i\gamma x \quad \ldots\ldots\ldots\ldots\ldots (G).$$
$$g_1(xi) = Gx.$$
$$G_1(xi) = gx.$$
$$\mathfrak{G}_1(xi) = \mathfrak{G}x.$$

Or otherwise,

$$\gamma(xi) = i\gamma_1 x \quad \ldots\ldots\ldots\ldots\ldots (H).$$
$$g(xi) = G_1 x.$$
$$G(xi) = g_1 x.$$
$$\mathfrak{G}(xi) = \mathfrak{G}_1 x,$$

equations which are useful in transforming almost any other property of the functions J.

The functions $J^\beta x$ are changed one into another, except as regards a constant multiplier, by the change of x into $x + \frac{\omega}{2}$.

This will be shown in a note, or it may be seen from some formulæ deduced immediately from the definitions of the functions $J_\beta x$, which will be given in the sequel.* Observing the relation between Jx and $J_\beta x$, we have in particular

$$\gamma\left(x+\frac{\omega}{2}\right)=\varepsilon^{\frac{1}{2}\beta\omega x}\,Agx \quad \ldots\ldots\ldots\ldots (G).$$

$$g\left(x+\frac{\omega}{2}\right)=\varepsilon^{\frac{1}{2}\omega\beta x}\,B\gamma x.$$

$$G\left(x+\frac{\omega}{2}\right)=\varepsilon^{\frac{1}{2}\beta\omega x}\,C\mathbb{G}x.$$

$$\mathbb{G}\left(x+\frac{\omega}{2}\right)=\varepsilon^{\frac{1}{2}\beta\omega x}\,DGx,$$

where $A, B, C, D,$ are most simply determined by writing $x=0,\ x=-\frac{\omega}{2}$. Putting at the same time $\varepsilon^{\beta\omega^2}=\varepsilon^{\frac{\pi\omega}{v}}=q_1^{-1}$,

$$A=\gamma\,\frac{\omega}{2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (H).$$

$$B=-\,q_1^{-\frac{1}{4}}\div\gamma\left(\frac{\omega}{2}\right).$$

$$C=G\left(\frac{\omega}{2}\right)=q_1^{-\frac{1}{4}}\div\mathbb{G}\,\frac{\omega}{2}.$$

$$D=\mathbb{G}\left(\frac{\omega}{2}\right)=q_1^{-\frac{1}{4}}\div G\,\frac{\omega}{2};$$

whence also

$$G\left(\frac{\omega}{2}\right)\mathbb{G}\left(\frac{\omega}{2}\right)=q_1^{-\frac{1}{4}} \quad \ldots\ldots\ldots\ldots (25).$$

Similarly, the functions $J_{-\beta}x$ are changed one into the other by the change of x into $x+\frac{1}{2}vi$. We have in the same way

$$\gamma\left(x+\frac{vi}{2}\right)=\varepsilon^{-\frac{1}{2}\beta vxi}\,A'Gx \quad \ldots\ldots\ldots\ldots (I).$$

$$g\left(x+\frac{vi}{2}\right)=\varepsilon^{-\frac{1}{2}\beta vxi}\,B'\mathbb{G}x.$$

$$G\left(x+\frac{vi}{2}\right)=\varepsilon^{-\frac{1}{2}\beta vxi}\,C'\gamma x.$$

$$\mathbb{G}\left(x+\frac{vi}{2}\right)=\varepsilon^{-\frac{1}{2}\beta vxi}\,D'gx.$$

* Not given in the present paper.

Whence $$A' = \gamma \frac{v i'}{2} \ldots\ldots\ldots\ldots\ldots\ldots (J).$$

$$B' = g \frac{v i'}{2} = q^{-\frac{1}{4}} \div \mathbb{G} \frac{v i'}{2},$$

$$C' = -q^{-\frac{1}{4}} \div \gamma \frac{v i'}{2},$$

$$D' = \mathbb{G} \frac{v i'}{2} = q^{-\frac{1}{4}} \div g \frac{v i'}{2};$$

where $\varepsilon^{\beta w^2} = \varepsilon^{\frac{\pi v}{\omega}} = q^{-1}$. It is obvious that the relation between q and q_1 is $lq \,.\, lq_1 = -\pi^2$.

We obtain from the above

$$g \frac{v i'}{2} \mathbb{G} \frac{v i'}{2} = q^{-\frac{1}{4}} \ldots\ldots\ldots (26).$$

Also, by making $x = \frac{v i'}{2}$ in the expression for $\gamma\left(x + \frac{\omega}{2}\right)$ and $x = \frac{\omega}{2}$ in that for $\gamma\left(x + \frac{v i'}{2}\right)$, we have

$$\gamma\left(\frac{\omega}{2}\right) g\left(\frac{v i'}{2}\right) = -i\,\gamma \frac{v i'}{2}\, G \frac{\omega}{2} \ldots\ldots (27),$$

and the same or an equivalent one would have been obtained from the functions g, G, $\mathbb{G}$.

By combining the above systems, we deduce one of the form

$$\gamma\left(x + \frac{\omega}{2} + \frac{v i'}{2}\right) = \varepsilon^{\frac{1}{2}\beta x (\omega - v i)} A' \mathbb{G} x \ldots\ldots (K),$$

$$g\left(x + \frac{\omega}{2} + \frac{v i'}{2}\right) = \varepsilon^{\frac{1}{2}\beta x (\omega - v i)} B' G x,$$

$$G\left(x + \frac{\omega}{2} + \frac{v i'}{2}\right) = \varepsilon^{\frac{1}{2}\beta x (\omega - v i)} C' g x,$$

$$\mathbb{G}\left(x + \frac{\omega}{2} + \frac{v i'}{2}\right) = \varepsilon^{\frac{1}{2}\beta x (\omega - v i)} D' \gamma x.$$

And observing the equation $\varepsilon^{\beta \omega v i} = \varepsilon^{\pi i} = (-1)$, with the following values for the coefficients,

$$A' = (-1)^{\frac{1}{4}} \,.\, \gamma \frac{\omega}{2}\, g \frac{v i'}{2} \ldots\ldots\ldots (L),$$

$$B' = -(-1)^{\frac{1}{4}} \,.\, q_1^{-\frac{1}{4}} \,.\, \gamma \frac{v i'}{2} \div \gamma \frac{\omega}{2},$$

$$C' = (-1)^{\frac{1}{4}} \cdot \mathbb{G}\,\frac{\omega}{2}\,\mathbb{G}\,\frac{\upsilon i}{2},$$

$$D' = -(-1)^{\frac{1}{4}} \cdot q^{-\frac{1}{4}} \cdot \mathbb{G}\,\frac{\omega}{2} \div \gamma\,\frac{\upsilon i}{2}.$$

Collecting the formulæ with connect $\gamma\left(\frac{\omega}{2}\right) \cdot \gamma\left(\frac{\upsilon i}{2}\right) \cdot \ldots$ these are

$$g\,\frac{\omega}{2} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (M).$$

$$G\,\frac{\upsilon i}{2} = 0,$$

$$G\,\frac{\omega}{2}\,\mathbb{G}\,\frac{\omega}{2} = q_1^{-\frac{1}{4}},$$

$$g\,\frac{\upsilon i}{2}\,\mathbb{G}\,\frac{\upsilon i}{2} = q^{-\frac{1}{4}},$$

$$\gamma\,\frac{\omega}{2} \cdot g\,\frac{\upsilon i}{2} = -i\gamma\left(\frac{\upsilon i}{2}\right) \cdot G\left(\frac{\omega}{2}\right).$$

And by the assistance of these

$$B'C' \div A'D' = B'D' \div A'C' = CD \div AB = -1 \quad \ldots\ldots\ldots (28),$$

$$A'B' \div C'D' = -A'B' \div C'D' = -\gamma^2\left(\frac{\upsilon i}{2}\right) \div \mathbb{G}^2\left(\frac{\upsilon i}{2}\right),$$

$$A'C' \div B'D' = -A'C' \div B'D' = \gamma^2\left(\frac{\omega}{2}\right) \div \mathbb{G}^2\left(\frac{\omega}{2}\right),$$

$$AD \div BC = -A'D' \div B'C' = \gamma^2\left(\frac{\omega}{2}\right) \div G^2\left(\frac{\omega}{2}\right) = -\gamma^2\left(\frac{\upsilon i}{2}\right) \div g^2\left(\frac{\upsilon i}{2}\right),$$

which will be required presently.

It is now easy to proceed to the general systems of formulæ,

$$\Theta = (-1)^{mn} \cdot \varepsilon^{\beta x \cdot (m\omega - n\upsilon i)} \cdot q_1^{-\frac{1}{2}m^2} \cdot q^{-\frac{1}{2}n^2} \ldots (M).$$

$$\gamma\{x + (m, n)\} = (-1)^{m+n} \cdot \Theta\gamma x,$$

$$g\{x + (m, n)\} = (-1)^{m} \cdot \Theta g x,$$

$$G\{x + (m, n)\} = (-1)^{n} \cdot \Theta G x,$$

$$\mathbb{G}\{x + (m, n)\} = \Theta\mathbb{G}x.$$

$$\Phi = (-1)^{n\left(m+\frac{1}{2}\right)} \cdot \varepsilon^{\beta x\left[\left(m+\frac{1}{2}\right)\omega - n\upsilon i\right]} q_1^{-\frac{1}{2}m^2 - \frac{1}{2}m} \cdot q^{-\frac{1}{2}n^2}.$$

$$\gamma\{x + (\overline{m}, n)\} = (-1)^{m+n} \cdot \Phi A g x,$$

$$g\{x + (\overline{m}, n)\} = (-1)^{m} \cdot \Phi B \gamma x,$$

$$G\{x + (\overline{m}, n)\} = (-1)^{n} \cdot \Phi C \mathbb{G} x,$$

$$\mathbb{G}\{x + (\overline{m}, n)\} = \Phi D G x.$$

$$\Psi = (-1)^{m(n+\frac{1}{2})} \cdot \varepsilon^{\beta x \cdot [m\omega - (n+\frac{1}{2})\nu i]} \, q_1^{-\frac{1}{2}m^2} \cdot q^{-\frac{1}{2}n^2 - \frac{1}{2}n}.$$

$$\gamma\{x + (m, \overline{n})\} = (-1)^{m+n} \cdot \Psi A' G x,$$
$$g\{x + (m, \overline{n})\} = (-1)^{m} \cdot \Psi B' \mathfrak{G} x,$$
$$G\{x + (m, \overline{n})\} = (-1)^{n} \cdot \Psi C' \gamma x,$$
$$\mathfrak{G}\{x + (m, \overline{n})\} = \Psi D' g x.$$

$$\Omega = (-1)^{mn + \frac{m}{2} + \frac{n}{2}} \cdot \varepsilon^{\beta x [(m+\frac{1}{2})\omega - (n+\frac{1}{2})\nu i]} \cdot q_1^{-\frac{1}{2}m^2 - \frac{1}{2}m} \cdot q^{-\frac{1}{2}n^2 - \frac{1}{2}n}.$$

$$\gamma\{x + (\overline{m}, \overline{n})\} = (-1)^{m+n} \cdot \Omega A' \mathfrak{G} x,$$
$$g\{x + (\overline{m}, \overline{n})\} = (-1)^{m} \cdot \Omega B' G x,$$
$$G\{x + (\overline{m}, \overline{n})\} = (-1)^{n} \cdot \Omega C' g x,$$
$$\mathfrak{G}\{x + (\overline{m}, \overline{n})\} = \cdot \Omega D' \gamma x.$$

Suppose $x = 0$, we have the new systems,

$$\Theta_0 = (-1)^{mn} \cdot q_1^{-\frac{1}{2}m^2} q^{-\frac{1}{2}n^2} \ldots\ldots (M bis).$$

$$\gamma(m, n) = 0, \qquad \gamma'(m, n) = (-1)^{m+n} \Theta_0,$$
$$g(m, n) = (-1)^{m} \cdot \Theta_0,$$
$$G(m, n) = (-1)^{n} \cdot \Theta_0,$$
$$\mathfrak{G}(m, n) = \Theta_0.$$

$$\Phi_0 = (-1)^{n \cdot (m+\frac{1}{2})} q_1^{-\frac{1}{2}m^2 - \frac{1}{2}m} q^{-\frac{1}{2}n^2}.$$

$$\gamma(\overline{m}, n) = (-1)^{m+n} \cdot \Phi_0 \cdot A,$$
$$g(\overline{m}, n) = 0, \qquad g'(\overline{m}, n) = (-1)^{m} \cdot \Phi_0 B,$$
$$G(\overline{m}, n) = (-1)^{n} \cdot \Phi_0 C,$$
$$\mathfrak{G}(\overline{m}, n) = \Phi_0 D.$$

$$\Psi_0 = (-1)^{m \cdot (n+\frac{1}{2})} q_1^{-\frac{1}{2}m^2} \cdot q^{-\frac{1}{2}n^2 - \frac{1}{2}n}.$$

$$\gamma(m, \overline{n}) = (-1)^{m+n} \cdot \Psi_0 A',$$
$$g(m, \overline{n}) = (-1)^{m} \cdot \Psi_0 B',$$
$$G(m, \overline{n}) = 0, \qquad G'(m, \overline{n}) = (-1)^{n} \cdot \Psi_0 C',$$
$$\mathfrak{G}(m, \overline{n}) = \Psi_0 D'.$$

$$\Omega_0 = (-1)^{mn + \frac{1}{2}m + \frac{1}{2}n} q_1^{-\frac{1}{2}m^2 - \frac{1}{2}m} \cdot q^{-\frac{1}{2}n^2 - \frac{1}{2}n}.$$

$$\gamma(\overline{m}, \overline{n}) = (-1)^{m+n} \cdot \Omega_0 A'',$$
$$g(\overline{m}, \overline{n}) = (-1)^{m} \cdot \Omega_0 B'',$$
$$G(\overline{m}, \overline{n}) = (-1)^{n} \cdot \Omega_0 C',$$
$$\mathfrak{G}(\overline{m}, \overline{n}) = 0, \qquad \mathfrak{G}'(m, \overline{n}) = \Omega_0 D'.$$

We obtain immediately, by taking the logarithmic differentials of the functions γx, gx, Gx, $\mathbb{G}x$, the equations

$$\gamma' x \div \gamma x = \Sigma\Sigma\{x-(m,n)\}^{-1}, \quad m=0, \ n=0 \text{ admissible},$$
$$g' x \div gx = \Sigma\Sigma\{x-(\overline{m},n)\}^{-1}, \ldots\ldots\ldots\ldots (N),$$
$$G' x \div Gx = \Sigma\Sigma\{x-(m,\overline{n})\}^{-1},$$
$$\mathbb{G}' x \div \mathbb{G}x = \Sigma\Sigma\{x-(\overline{m},\overline{n})\}^{-1},$$

the limits being the same as in the case of the factorial expressions.

Consider an equation

$$gx.Gx \div \gamma x.\mathbb{G}x = \Sigma\Sigma[\mathfrak{A}\{x-(m,n)\}^{-1}+\mathfrak{B}\{x-(\overline{m},\overline{n})\}^{-1}] \ldots (29),$$

we have

$$\mathfrak{A} = g(m,n)G(m,n) \div \gamma'(m,n)\mathbb{G}(m,n) = 1 \ldots\ldots\ldots\ldots (30),$$
$$\mathfrak{B} = g(\overline{m},\overline{n})G(\overline{m},\overline{n}) \div \gamma(\overline{m},\overline{n})\mathbb{G}'(\overline{m},\overline{n}) = B''C'' \div A''D'' = -1 \ldots (31).$$

(The application of the ordinary method of decomposition into partial fractions, which is in general exceedingly precarious when applied to transcendental functions, is justified here by a theorem of Cauchy's, which will presently be quoted.) We have thus

$$gx\,Gx \div \gamma x\,\mathbb{G}x = (\gamma' x \div \gamma x) - (\mathbb{G}' x \div \mathbb{G}x),$$

and similarly

$$gx\,\mathbb{G}x \div \gamma x\,Gx = (\gamma' x \div \gamma x) - (G' x \div Gx), \ldots\ldots (O),$$
$$Gx\,\mathbb{G}x \div \gamma x\,gx = (\gamma' x \div \gamma x) - (g' x \div gx),$$
$$-b^2\gamma x\,\mathbb{G}x \div gx\,Gx = (g' x \div gx) - (G' x \div Gx),$$
$$e^2\gamma x\,gx \div Gx\,\mathbb{G}x = (G' x \div Gx) - (\mathbb{G}' x \div \mathbb{G}x),$$
$$c^2\gamma x\,Gx \div \mathbb{G}x\,gx = (\mathbb{G}' x \div \mathbb{G}x) - (g' x \div gx);$$

in which we have written

$$\gamma\frac{vi}{2} \div \mathbb{G}\left(\frac{vi}{2}\right) = \frac{i}{c} \ldots\ldots\ldots\ldots (32),$$
$$\gamma\left(\frac{\omega}{2}\right) \div \mathbb{G}\left(\frac{\omega}{2}\right) = \frac{1}{c},$$
$$\gamma\left(\frac{\omega}{2}\right) \div G\left(\frac{\omega}{2}\right) = -i\left(\gamma\frac{vi}{2} \div g\frac{vi}{2}\right) = \frac{1}{b}.$$

Eliminating the derived coefficients,

$$G^2 x - \mathbb{G}^2 x = e^2\gamma^2 x, \ldots\ldots\ldots (33),$$
$$g^2 x - G^2 x = -b^2\gamma^2 x,$$
$$\mathbb{G}^2 x - g^2 x = c^2\gamma^2 x.$$

Adding $b^2 = e^2 + c^2$ or $b = \sqrt{(e^2 + c^2)}$, in which sense it will continue to be used.

Also,
$$g^2x = \mathbb{G}^2x - c^2\gamma^2x, \quad \ldots\ldots\ldots (P)$$
$$G^2x = \mathbb{G}^2x + e^2\gamma^2x.$$

Suppose
$$\phi x = \gamma x \div \mathbb{G}x, \quad \ldots\ldots\ldots\ldots (Q).$$
$$fx = gx \div \mathbb{G}x,$$
$$Fx = Gx \div \mathbb{G}x.$$

Then
$$f^2x = 1 - c^2\phi^2x, \quad \ldots\ldots\ldots\ldots (R),$$
$$F^2x = 1 + e^2\phi^2x,$$

and also
$$\phi'x = fx\,Fx, \quad \ldots\ldots\ldots\ldots (S),$$
$$f'x = -c^2\phi x\,Fx,$$
$$F'x = e^2\phi x\,fx.$$

Whence, putting for fx, Fx, their values

$$1 = \frac{\phi'x}{\sqrt{(1 - c^2\phi^2x)\,(1 + e^2\phi^2x)}} \quad \ldots\ldots (T);$$

or writing $\phi x = y$, and integrating,

$$x = \int_0^{\phi x} \frac{dy}{\sqrt{(1 - c^2y^2)\,(1 + e^2y^2)}} \quad \ldots\ldots (U),$$

or
$$\phi^{-1}y = \int_0^{y} \frac{dy}{\sqrt{(1 - c^2y^2)\,(1 + e^2y^2)}},$$

which shows that ϕ is an inverse elliptic function.

The equations which are the foundation of the theory of the functions ϕ, f, F, are deduced immediately from the equations (S). (*Abel Œuvres*, tom. I. p. 143.) These are

$$\phi(x + y) = \frac{\phi x\,fy\,Fy + \phi y\,fx\,Fx}{1 + e^2c^2\,\phi^2x\,\phi^2y} \quad \ldots\ldots (V),$$

$$f(x + y) = \frac{fx\,fy - c^2\,\phi x\,\phi y\,Fx\,Fy}{1 + e^2c^2\,\phi^2x\,\phi^2y},$$

$$F(x + y) = \frac{Fx\,Fy + e^2\,\phi x\,\phi y\,fx\,fy}{1 + e^2c^2\,\phi^2x\,\phi^2y};$$

so that from this point we may take for granted any properties of these functions. We see, for instance, immediately, $\phi\,\frac{\omega i}{2} = \frac{i}{e}$, $\phi\left(\frac{\omega}{2}\right) = \frac{1}{c}$; whence

$$\frac{\omega}{2} = \int_0^{\frac{1}{c}} \frac{dy}{\sqrt{(1 - c^2y^2)\,(1 + e^2y^2)}}, \quad \ldots\ldots (W),$$

$$\frac{v i}{2} = \int_0^{\frac{i}{e}} \frac{dy}{\sqrt{(1-c^2y^2)(1+e^2y^2)}}, \quad \text{or} \quad \frac{v}{2} = \int_0^{\frac{1}{e}} \frac{dy}{\sqrt{(1+c^2y^2)(1-e^2y^2)}} \quad \ldots\ldots (X),$$

which give the values of ω, v in terms of c, e; values which may be developed in a variety of ways, in infinite series. We may also express $\gamma \frac{\omega}{2}$, &c., and consequently A, B...&c., by means of the quantities c, e. We have only to combine the equations

$$\gamma\left(\frac{\omega}{2}\right) \div \mathbb{G}\left(\frac{\omega}{2}\right) = \frac{1}{c}, \quad \gamma\left(\frac{v i}{2}\right) \div \mathbb{G}\,\frac{v i}{2} = \frac{i}{e},$$

$$G\,\frac{\omega}{2} \div \mathbb{G}\,\frac{\omega}{2} = \frac{b}{c}, \quad g\,\frac{v i}{2} \div \mathbb{G}\,\frac{v i}{2} = \frac{b}{e} \ldots\ldots (34),$$

with the former relations between these quantities, and we have

$$\gamma\,\frac{\omega}{2} = b^{-\frac{1}{2}} c^{-\frac{1}{2}} q_1^{-\frac{1}{8}}, \qquad \gamma\,\frac{v i}{2} = i b^{-\frac{1}{2}} e^{-\frac{1}{2}} q^{-\frac{1}{8}}, \ldots\ldots (Y),$$

$$g\,\frac{\omega}{2} = 0, \qquad g\,\frac{v i}{2} = b^{\frac{1}{2}} e^{-\frac{1}{2}} q^{-\frac{1}{8}}.$$

$$G\,\frac{\omega}{2} = b^{\frac{1}{2}} c^{-\frac{1}{2}} q_1^{-\frac{1}{8}}, \qquad \mathbb{G}\,\frac{v i}{2} = 0,$$

$$\mathbb{G}\,\frac{\omega}{2} = b^{-\frac{1}{2}} c^{\frac{1}{2}} q_1^{-\frac{1}{8}}, \qquad \mathbb{G}\,\frac{v i}{2} = b^{-\frac{1}{2}} e^{\frac{1}{2}} q^{-\frac{1}{8}}.$$

$$A = b^{-\frac{1}{2}} c^{-\frac{1}{2}} q_1^{-\frac{1}{8}}, \quad A' = i b^{-\frac{1}{2}} e^{-\frac{1}{2}} q^{-\frac{1}{8}}, \quad A'' = (-1)^{\frac{1}{4}} c^{-\frac{1}{2}} e^{-\frac{1}{2}} q_1^{-\frac{1}{8}} q^{-\frac{1}{8}}, \quad \ldots\ldots (Z).$$

$$B = -b^{\frac{1}{2}} c^{\frac{1}{2}} q_1^{-\frac{1}{8}}, \quad B' = b^{\frac{1}{2}} e^{-\frac{1}{2}} q^{-\frac{1}{8}}, \quad B'' = -(-1)^{\frac{1}{4}} i c^{\frac{1}{2}} e^{-\frac{1}{2}} q_1^{-\frac{1}{8}} q^{-\frac{1}{8}},$$

$$C = b^{\frac{1}{2}} c^{-\frac{1}{2}} q_1^{-\frac{1}{8}}, \quad C' = -i b^{\frac{1}{2}} e^{\frac{1}{2}} q^{-\frac{1}{8}}, \quad C'' = (-1)^{\frac{1}{4}} c^{-\frac{1}{2}} e^{\frac{1}{2}} q_1^{-\frac{1}{8}} q^{-\frac{1}{8}},$$

$$D = b^{-\frac{1}{2}} c^{\frac{1}{2}} q_1^{-\frac{1}{8}}, \quad D' = b^{-\frac{1}{2}} e^{\frac{1}{2}} q^{-\frac{1}{8}}, \quad D'' = -(-1)^{\frac{1}{4}} i c^{\frac{1}{2}} e^{\frac{1}{2}} q_1^{-\frac{1}{8}} q^{-\frac{1}{8}},$$

which are to be substituted in any formulæ into which these quantities enter.

The following is Cauchy's Theorem, (*Exercises de Math.* tom. II. p. 289).

"If in attributing to the modulus r of the variable

$$z = r\{\cos p + \sqrt{(-1)} \sin p\} \ldots\ldots (35),$$

infinitely great values, these can be chosen so that the two functions

$$\frac{fz+f(-z)}{2}, \quad \frac{fz-f(-z)}{2z}, \ldots\ldots (36),$$

sensibly vanish, whatever be the value of p, or vanish in general, though ceasing to do so, and obtaining *finite* values for certain particular values of p; then

$$fx = \mathfrak{C} \frac{\{(fz)\}}{x-z} \ldots\ldots\ldots (37),$$

the integral residue being reduced to its principal value."

To understand this, it is only necessary to remark that the integral residue in question is the series of fractions that would be obtained by the ordinary process of decomposition; and by the principal value is meant, that *all* those roots are to be taken, the modulus of which is not greater than a certain limit, this limit being afterwards made infinite.

Suppose now fx is a fraction, the numerator and denominator of which are monomials of the form $(\gamma x)^l (gx)^m \ldots$, $l, m \ldots$ being positive integers, and of course no common factor being left in the numerator and denominator.

Let λ be the excess of the degree of the denominator over that of the numerator. Suppose the modulus r of (z) has any value not the same with any of the moduli of

$$(m, n), \ (\overline{m}, n), \ (m, \overline{n}), \ (\overline{m}, \overline{n}) \ldots . (38).$$

Then we have

$$r(\cos p + i \sin p) = m\omega + n\nu i + \theta \ldots (39),$$

θ being a finite quantity, such that none of the functions $J\theta$ vanish. m and n are the greatest integer values which allow the possible part of θ and the coefficient of its impossible part to remain positive. We have therefore

$$m^2\omega^2 + n^2\nu^2 = r^2 - M \ldots\ldots\ldots (40),$$

M being finite; or when r is infinite, at least one of the values m, n is infinite. The function fz reduces itself to the form

$$q_1^{\frac{\lambda m^2}{2}} q^{\frac{\lambda n^2}{2}} \varepsilon^{mA+nB} F \ldots\ldots\ldots (41),$$

where F is finite. Hence q_1 and q being always less than unity, fz, and consequently both $\frac{1}{2}\{fz+f(-z)\}$ and $\frac{1}{2z}\{fz-f(-z)\}$ vanish for $r = \infty$, as long as λ is positive.

In the case of $\lambda = 0$, the conditions are still satisfied, if we suppose fx to denote an uneven function of x: for when $\lambda=0$, the index of exponential in the above expression vanishes,

or fz is constantly finite. But fz being an odd function of z, $fz + f(-z) = 0$. And $\frac{1}{2z}\{fz - f(-z)\}$ vanishes for z infinite, on account of the z in the denominator: hence the expansion is admissible in this case. But it is certainly so also, in a great many cases at least, where fz is an even function of z; for these may be deduced from the others by a simple change in the value of the variable. For instance, from the expansion of $\gamma x \div gx$, which is an odd function, by writing $x + \frac{\upsilon i}{2}$ for x, we obtain that of $Gx \div \mathfrak{G}x$, which is even.

A case of some importance is when the function is of the above form, multiplied by an exponential $\varepsilon^{\frac{1}{2}ax^2+bx}$. Here writing $z = m\omega + n\upsilon i + \theta$, the admissibility of the formula depends on the evanescence of

$$\varepsilon^{\frac{1}{2}a(m\omega+n\upsilon i)^2} q_1^{\frac{1}{2}\lambda m^2} q^{\frac{1}{2}\lambda n^2} \quad \ldots\ldots (42);$$

or, if $a = h + ki$, this becomes, omitting a finite factor,

$$\varepsilon^{-\frac{1}{2}m^2(\lambda\beta - h)\omega^2 - \frac{1}{2}n^2(\lambda\beta + h)\upsilon^2 - kmn\omega\upsilon} \quad \ldots (43),$$

which vanishes if $h^2 + k^2 < \lambda^2\beta^2$, i.e. the modulus of (a) is less than $\lambda\beta$. The limiting case is admissible when the series is convergent.

We obtain in this way a very great variety of formulæ. For instance,

$$\varepsilon^{\frac{1}{2}ax^2+bx} \div \gamma x$$
$$= \Sigma\Sigma\left[(-1)^{-mn-m-n}\, \varepsilon^{\frac{1}{2}a(m,n)^2+b(m,n)} q_1^{\frac{1}{2}m^2} q^{\frac{1}{2}n^2} \{x-(m,n)\}^{-1}\right] \ldots (A'),$$

$$\varepsilon^{\frac{1}{2}ax^2+bx} \div gx$$
$$= -b^{\frac{1}{2}} c^{-\frac{1}{2}} \Sigma\Sigma\left[(-1)^{-mn-m-\frac{1}{2}n}\, \varepsilon^{\frac{1}{2}a(\overline{m},n)^2+b(\overline{m},n)} q_1^{\frac{1}{2}(m+\frac{1}{2})^2} q^{\frac{1}{2}n^2} \{x-(\overline{m},n)\}^{-1}\right],$$

$$\varepsilon^{\frac{1}{2}ax^2+bx} \div Gx$$
$$= ib^{-\frac{1}{2}} e^{-\frac{1}{2}} \Sigma\Sigma\left[(-1)^{-mn-\frac{1}{2}m-n}\, \varepsilon^{\frac{1}{2}a(m,n)^2+b(m,\overline{n})} q_1^{\frac{1}{2}m^2} q^{\frac{1}{2}(n+\frac{1}{2})^2} \{x-(m,\overline{n})\}^{-1}\right],$$

$$\varepsilon^{\frac{1}{2}ax^2+bx} \div \mathfrak{G}x$$
$$= ic^{-\frac{1}{2}} e^{-\frac{1}{2}} \Sigma\Sigma\left[(-1)^{-(m+\frac{1}{2})(n+\frac{1}{2})}\, \varepsilon^{\frac{1}{2}a(\overline{m},\overline{n})^2+b(\overline{m},\overline{n})} q_1^{\frac{1}{2}(m+\frac{1}{2})^2} q^{\frac{1}{2}(n+\frac{1}{2})^2} \{x-(\overline{m},\overline{n})\}^{-1}\right],$$

in which the modulus of a must not exceed β: in the limiting cases, for $a = \beta$, b must be entirely impossible, and for $a = -\beta$, b must be entirely real. The formulæ for γx are

$$\varepsilon^{\frac{1}{2}\beta x^2+bx} \div \gamma x = \Sigma\Sigma(-1)^{-m-n} q^{n^2} \varepsilon^{b(m,n)} \{x-(m,n)\}^{-1} \ldots (44),$$

$$\varepsilon^{-\frac{1}{2}\beta x^2+bx} \div \gamma x = \Sigma\Sigma(-1)^{m-n} q_1^{m^2} \varepsilon^{b(m,n)} \{x-(m,n)\}^{-1};$$

and for $b = 0$,

$$\varepsilon^{\frac{1}{2}\beta x^2} \div \gamma x = \Sigma\Sigma(-1)^{-m-n} q^{n^2} \{x-(m,n)\}^{-1} \ldots\ldots (45),$$

$$\varepsilon^{-\frac{1}{2}\beta x^2} \div \gamma x = \Sigma\Sigma(-1)^{-m-n} q_1^{m^2} \{x-(m,n)\}^{-1}.$$

Next the system,

$$\mathfrak{G}x \div \gamma x = \Sigma\Sigma\,(-1)^{m+n}\,\{x-(m,n)\}^{-1} \ldots\ldots (B').$$
$$gx \div \gamma x = \Sigma\Sigma\,(-1)^{m}\,\{x-(m,n)^{-1},$$
$$\mathfrak{G}x \div \gamma x = \Sigma\Sigma\,(-1)^{n}\,\{x-(m,n)\}^{-1};$$

$$\gamma x \div gx = -\,b^{-1}\,c^{-1}\,\Sigma\Sigma\,(-1)^{n}\,\{x-(\bar{m},n)\}^{-1},$$
$$Gx \div gx = -\,c^{-1}\,\Sigma\Sigma\,(-1)^{m+n}\,\{x-(\bar{m},n)\}^{-1},$$
$$\mathfrak{G}x \div gx = -\,b^{-1}\,\Sigma\Sigma\,(-1)^{m}\,\{x-(\bar{m},n)\}^{-1};$$

$$\gamma x \div Gx = -\,b^{-1}\,e^{-1}\,\Sigma\Sigma\,(-1)^{n}\,\{x-(m,\bar{n})\}^{-1},$$
$$gx \div Gx = \;\;i\,e^{-1}\,\Sigma\Sigma\,(-1)^{m+n}\,\{x-(m,\bar{n})\}^{-1},$$
$$\mathfrak{G}x \div Gx = \;\;i\,b^{-1}\,\Sigma\Sigma\,(-1)^{n}\,\{x-(m,\bar{n})\}^{-1};$$

$$\gamma x \div \mathfrak{G}x = \;\;i\,c^{-1}\,e^{-1}\,\Sigma\Sigma\,(-1)^{m+n}\,\{x-(\bar{m},\bar{n})\}^{-1},$$
$$gx \div \mathfrak{G}x = \;\;\;e^{-1}\;\;\Sigma\Sigma\,(-1)^{n}\,\{x-(\bar{m},\bar{n})\}^{-1},$$
$$Gx \div \mathfrak{G}x = \;\;i\,c^{-1}\;\;\Sigma\Sigma\,(-1)^{n}\,\{x-(\bar{m},\bar{n})\}^{-1}.$$

which is partially given by Abel.

We may obtain, in like manner, expressions for the functions

$$\frac{1}{\gamma x\,gx},\ \frac{1}{\gamma x\,Gx},\ \ldots \text{(six terms of this form)} \ldots\ (C'),$$

$$\frac{Gx}{\gamma x\,gx},\ \ldots \qquad \text{(twelve)} \qquad \ldots\ (D'),$$

$$\frac{\gamma x\,gx}{\mathfrak{G}x\,Gx},\ \ldots \qquad \text{(six)} \qquad \ldots\ (E'),$$

$$\frac{1}{\gamma x\,gx\,Gx},\ \ldots \qquad \text{(four)} \qquad \ldots\ (F'),$$

$$\frac{\mathfrak{G}x}{\gamma x\,gx\,Gx},\ \ldots \qquad \text{(four)} \qquad \ldots\ (G'),$$

$$\frac{1}{\gamma x\,gx\,Gx\,\mathfrak{G}x},\ \ldots \qquad \text{(one)} \qquad \ldots\ (H');$$

each of them, except (E'), (the system for which, admitting no exponential, has already been given,) multiplied by an exponential $\varepsilon^{\lambda(ax+bx}$, the limits of (a) being $\pm 2\beta$, $\pm\beta$, $-$, $\pm 3\beta$, $\pm 2\beta$, $\pm 4\beta$. For the limiting values, b must be entirely impossible for the superior limit, and entirely possible for the inferior one.

Thus the last case is

$$\frac{1}{\gamma x\, gx\, Gx\, \mathfrak{G}x}\, \varepsilon^{\frac{1}{2}\alpha x^2+bx} \quad \ldots\ldots\ldots\ldots (H'),$$

$$= \Sigma\Sigma\left[\varepsilon^{\frac{1}{2}\alpha(m,n)^2+b(m,n)}\, q_1^{2m^2}\, q^{2n^2}\, \{x-(m,n)\}^{-1}\right]$$

$$- \Sigma\Sigma\left[\varepsilon^{\frac{1}{2}\alpha(\bar{m},n)^2+b(\bar{m},n)}\, q_1^{2(m+\frac{1}{2})^2}\, q^{2n^2}\, \{x-(\bar{m},n)\}^{-1}\right]$$

$$+ \Sigma\Sigma\left[\varepsilon^{\frac{1}{2}\alpha(m,\bar{n})^2+b(m,\bar{n})}\, q_1^{2m^2}\, q^{2(n+\frac{1}{2})^2}\, \{x-(m,\bar{n})\}^{-1}\right]$$

$$- \Sigma\Sigma\left[\varepsilon^{\frac{1}{2}\alpha(\bar{m},\bar{n})^2+b(\bar{m},\bar{n})}\, q_1^{2(m+\frac{1}{2})^2}\, q^{2(n+\frac{1}{2})^2}\, \{x-(\bar{m},\bar{n})\}^{-1}\right];$$

in particular

$$\frac{1}{\gamma x\, gx\, Gx\, \mathfrak{G}x}\, \varepsilon^{2\beta x^2} = \Sigma\Sigma\, q_1^{4m^2}\, \{x-(m,n)\}^{-1} \ldots . (46),$$

$$- \Sigma\Sigma\, q_1^{(2m+1)^2}\, \{x-(\bar{m},n)\}^{-1}$$

$$+ \Sigma\Sigma\, q_1^{4m^2}\, \{x-(\bar{m},\bar{n})\}^{-1}$$

$$+ \Sigma\Sigma\, q_1^{(2m+1)^2}\, \{x-(\bar{m},\bar{n})\}^{-1},$$

or the analogous formula obtained by changing β, q_1, m into $-\beta$, q, n.

The function $\phi^2 x$, which is even, and for which $\lambda = 0$, cannot be expanded entirely in a series of partial fractions; but $(x-a)^{-1}\phi^2 x$ may be so expanded. Multiply by $(x-a)$, the second side has for its general term

$$(x-a)\,(Mx+N)\,\{x-(\bar{m},\bar{n})^{-2},$$

equivalent to

$$K' + (M'x+N')\,\{x-(\bar{m},\bar{n})\}^{-2}.$$

Summing all the K''s, we have an equation of the form

$$\phi^2 x = A + \Sigma\Sigma\left[L\,\{x-(\bar{m},\bar{n})\}^{-2} + M\,\{x-(\bar{m},\bar{n})\}^{-1}\right] \ldots (47).$$

To determine the coefficients as simply as possible, change x into $x+\frac{1}{2}\omega+\frac{1}{2}n\omega i$,

$$-e^{-2}c^{-2}\,\overline{\phi x}\big|^{-2} = A + \Sigma\Sigma\left[L\,\{x-(m,n)\}^{-2} + M\,\{x-(m,n)\}^{-1}\right]$$

$$\ldots, (48),$$

$$L = -e^{-2}c^{-2}\left[\{x-(m,n)\}^2\,\overline{\phi x}\big|^{-2}\right] \ldots\ldots\ldots\ldots (49),$$

$$M = -e^{-2}c^{-2}\,d_x\left[\{x-(m,n)\}^2\,\overline{\phi x}\big|^{-2}\right], \quad x=(m,n).$$

Or writing $x+(m,n)$ for x,

$$L = -e^{-2}c^{-2}\,(x^2\overline{\phi x}\big|^{-2}), \quad (x=0), \qquad L = e^{-2}c^{-2} \ldots (50),$$

$$M = -e^{-2}c^{-2}\,d_x\,(x^2\overline{\phi x}\big|^{-2}) = 0;$$

$$\phi^2 x = A - e^{-2}c^{-2}\,\Sigma\Sigma\,\{x-(\bar{m},\bar{n})\}^{-2} \ldots\ldots\ldots (51).$$

Integrating twice,

$$\int_0 dx \int_0 dx \,.\, \phi^2 x = \tfrac{1}{2} A x^2 + e^{-2} c^{-2} \Sigma\Sigma\, l \{x - (\overline{m}, \overline{n})\} \ldots (52),$$

or
$$\mathfrak{G}x = \varepsilon^{-\frac{1}{6} e^2 c^2 A x^3 + e^2 c^2} \int_0 dx \int_0 dx \,.\, \phi^2 x \ldots\ldots (53),$$

an equation from which it is easy to determine the coefficient A.

Suppose for a moment $\phi_1 x = \int_0 \phi^2 x\, dx$, $\phi_2 x = \int_0 \phi_1 x\, dx$; then, since $\phi^2(x+\omega) - \phi^2 x = 0$,

$$\phi_1(x+\omega) = \phi_1 x = \phi_1 \omega, \quad \phi_2(x+\omega) - \phi_2 x = \phi_2 \omega + x \phi_1 \omega.$$

But similarly $\phi^2 x - \phi^2(\omega - x) = 0$; whence

$$\phi_1 x + \phi_1(\omega - x) = \phi_1 \omega, \quad \phi_2 x - \phi_2(\omega - x) + \phi_2 \omega = x \phi_1 \omega;$$

whence, writing $x = \frac{\omega}{2}$,

$$\phi_1 \omega = 2\phi_1 \frac{\omega}{2}, \quad \phi_2 \omega = \omega . \phi_1 \frac{\omega}{2}, \text{ or } \phi_2(x+\omega) - \phi_2 x = \phi_1\left(\frac{\omega}{2}\right)(2x+\omega).$$

Hence
$$\mathfrak{G}(x+\omega) = \varepsilon^{-\frac{1}{6} e^2 c^2 \left(A - \frac{\omega}{2} \phi_1 \frac{\omega}{2}\right)(2\omega x + \omega^2)} \mathfrak{G}x \ldots . (54).$$

But
$$\mathfrak{G}(x+\omega) = \epsilon^{\beta\omega x} q_1^{-\frac{1}{6}} \mathfrak{G}x = \epsilon^{\frac{\beta}{2}(2\omega x + \omega^2)} \mathfrak{G}x \ldots . (55);$$

or, comparing these,

$$-e^2 c^2 \left(A - \frac{2}{\omega} \phi_1 \frac{\omega}{2}\right) = \beta \ldots\ldots\ldots (56),$$

$$-\tfrac{1}{2} e^2 c^2 A = \tfrac{1}{2}\beta - \frac{e^2 c^2}{\omega} \phi_1 \frac{\omega}{2} \ldots\ldots\ldots (57),$$

or writing

$$M = \frac{e^2 c^2}{\omega} \int_0^{\frac{\omega}{2}} \phi^2 x\, dx \ldots\ldots\ldots\ldots (58),$$

$$\mathfrak{G}x = \varepsilon^{(\frac{1}{2}\beta - M) x^2 + e^2 c^2 \int_0 dx \int_0 dx \,.\, \phi^2 x} \ldots\ldots (I');$$

which is the formulæ corresponding to the one of Jacobi's referred to at the beginning of this paper. Analogous formulæ may be deduced from it by writing $x + \frac{\omega}{2}$, or $x + \frac{\upsilon i}{2}$, or $x + \frac{\omega}{2} + \frac{\upsilon i}{2}$, instead of x.

The following formulæ, making the necessary changes of notation, are taken from Jacobi. We have

$$\phi^2(x+a) - \phi^2(x-a) = \frac{4\phi a . fa . Fa . \phi x . fx Fx}{(1 + e^2 c^2 . \phi^2 a . \phi^2 x)^2} \ldots\ldots (59),$$

whence

$$\int_0 \{\phi^2(x+a) - \phi^2 . (x-a)\}\, dx = \frac{2\phi a . fa . Fa . \phi^2 x}{1 + e^2 c^2 \phi^2 . a . \phi^2 x} \ldots (60).$$

The first side of which is

$$\int_{-a}.\phi^2.(x+a)\,dx - \int_a.\phi^2\,(x-a)\,dx - 2.\int_0\phi^2 a.da \ldots (61).$$

Hence, multiplying by e^2c^2, and observing the value of $\mathbb{G}x$,

$$\frac{\mathbb{G}'\,(x+a)}{\mathbb{G}\,(x+a)} - \frac{\mathbb{G}'\,(x-a)}{\mathbb{G}\,(x-a)} - 2\,\frac{\mathbb{G}'a}{\mathbb{G}a} = \frac{2e^2c^2.fa.Fa.\phi a.\phi^2 x}{1+e^2c^2\phi^2 a.\phi^2 x} \ldots (62).$$

If in this case we interchange x, a and $a^d d$,

$$\frac{\mathbb{G}'x}{\mathbb{G}x} + \frac{\mathbb{G}'a}{\mathbb{G}a} - \frac{\mathbb{G}'\,(x+a)}{\mathbb{G}\,(x+a)} = e^2c^2.\phi a.\phi x.\phi\,(a+x) \ldots (63).$$

[By subtracting, we should have obtained an equation only differing from the above in the sign of (a)].

Integrating the last equation but one, with respect to (a),

$$l\mathbb{G}\,(x+a) + l\mathbb{G}\,(x-a) - 2l\mathbb{G}x - 2l\mathbb{G}a = l\,(1+e^2c^2\phi^2 x\phi^2 a).$$

the integral being taken from $a=0$. Whence

$$\mathbb{G}\,(x+a)\,\mathbb{G}\,(x-a) = \mathbb{G}^2x\mathbb{G}^2a\,(1+e^2c^2\phi^2 x\phi^2 a) \ldots (64).$$

Or $\mathbb{G}\,(x+a)\,\mathbb{G}\,(x-a\,) = \mathbb{G}^2x.\mathbb{G}^2a + e^2c^2.\gamma^2x\gamma^2a,$

whence also

$$\left.\begin{aligned} \gamma\,(x+a)\,\gamma\,(x-a) &= \gamma^2x\mathbb{G}^2a - \gamma^2.a\mathbb{G}^2x. \\ g\,(x+a)\,g\,(x-a) &= g^2x\mathbb{G}^2x - c^2.g^2a.\mathbb{G}^2x. \\ G\,(x+a).\,G\,(x-a) &= G^2x.\mathbb{G}^2a + e^2G^2a.\mathbb{G}^2x. \end{aligned}\right\} (J').$$

These equations being obtained from the first by the change of x into $x+\frac{\omega}{2}$, $x+\frac{vi}{2}$, $x+\frac{\omega}{2}+\frac{vi}{2}$. They form a most important group of formulæ in the present theory. (By integrating the same formulæ with respect to x, and representing by $\Pi(x,a)$ the integral $\int_0 \frac{-e^2c^2\phi afaFa\phi^2 x dx}{1+e^2c^2\phi^2 a\phi^2 x}$, Jacobi obtains

$$\Pi(x,a) = \tfrac{1}{2}l\,\frac{\mathbb{G}\,(x-a)}{\mathbb{G}\,(x+a)} + x\,\frac{\mathbb{G}'a}{\mathbb{G}a}:$$

an equation which conducts him almost immediately to the formulæ for the addition of the argument or of the parameter in the function Π. This, however, is not very closely connected with the present subject. For some formulæ also deduced from (63), by which $\frac{\mathbb{G}\,(x-a)\,\mathbb{G}\,(y-a)\,\mathbb{G}\,(x+y+a)}{\mathbb{G}\,(x+a)\,\mathbb{G}\,(y+a)\,\mathbb{G}\,(x+y-a)}$ is expressed in terms of the function ϕ. See Jacobi.

NOTE.—We have

$$\gamma_\beta x = x\Pi\Pi\left(1 + \frac{x}{(m, n)}\right).$$

$$g_\beta x = \Pi\Pi\left(1 + \frac{x}{(m, n)}\right).$$

the limits of n being $\pm q$, and those of m being $\pm p$, in the first case, and $p, -p-1$, in the second case. Also $\frac{p}{q} = \infty$. We deduce immediately

$$\gamma_\beta\left(x + \frac{\omega}{2}\right) = \left(x + \frac{\omega}{2}\right)\Pi\Pi\left\{1 + \frac{\left(x + \frac{\omega}{2}\right)}{(m, n)}\right\}$$

$$= \Pi\Pi.\left(1 + \frac{x}{(m, n)}\right) \div \frac{\omega}{2}\ \Pi\Pi\frac{(m, n)}{(m, n)};$$

(paying attention to the omission of $(m=0,\ n=0)$ in $\gamma_\beta x$, and supposing that this value enters into the numerator of the expression just obtained, but not into its denominator). This is of the form

$$\gamma_\beta\left(x + \frac{\omega}{2}\right) = A\Pi\Pi\left(1 + \frac{x}{(m, n)}\right);$$

but the limits are not the same in this product and in $g_\beta x$. In the latter m assumes the value $-p-1$, which it does not in the former. Hence

$$\gamma_\beta\left(x + \frac{\omega}{2}\right) \div g_\beta x = A \div \Pi_n\left(1 + \frac{x}{-(p + \frac{1}{2})\,\omega + n\nu i}\right).$$

And the above product reduces itself to unity in consequence of all the values assumed by n being indefinitely small compared with the quantity $(p + \frac{1}{2})$; we have therefore

$$\gamma_\beta\left(x + \frac{\omega}{2}\right) = Ag_\beta x \ldots\ldots\ldots\ldots (27),$$

and similar expressions for the remaining functions. To illustrate this further, suppose we had been considering, instead of $\gamma_\beta x$ the function $\gamma_{-\beta} x$, given by the same formulæ, only instead of $\frac{p}{q} = \infty$, $\frac{p}{q} = 0$. We have in this case also

$$\gamma_{-\beta}\left(x + \frac{\omega}{2}\right) \div g_{-\beta} x = A' \div \Pi_n\left(1 + \frac{x}{(-p + \frac{1}{2})\,\omega + n\nu i}\right).$$

A' different from A on account of the different limits. The divisor of the second side takes the form

$$\{x-(p+\tfrac{1}{2})\omega.\}.\Pi\left(1+\frac{x-(p+\frac{1}{2})\omega}{n\nu i}\right)\div(-p+\tfrac{1}{2})\omega.\Pi\left(1-\frac{(p+\frac{1}{2})\omega}{n\nu i}\right),$$

and the extreme values of n being infinite as compared with p. This may be reduced to

$$-\sin\frac{\pi}{\nu i}\{x-(p+\tfrac{1}{2})\omega\}\div\sin(p+\tfrac{1}{2})\omega.$$

$$=\varepsilon^{\frac{\pi}{\nu}[(p+\frac{1}{2})\omega-x]}\div\varepsilon^{\frac{\pi}{\nu}[(p+\frac{1}{2})\omega]}=\varepsilon^{-\frac{\pi x}{\nu}};$$

neglecting the exponentials whose indices are infinitely great and negative. Observing the value of β this becomes $\varepsilon^{-\omega\beta x}$, and we have

$$\gamma_{-\beta}\left(x+\frac{\omega}{2}\right)=\varepsilon^{\beta\omega x}.A'g_{-\beta}.x:$$

a result of the form of that which would be deduced from the equations $\gamma_{-\beta}x=\varepsilon^{\beta x^2}\gamma_\beta x$, $g_{-\beta}x=\varepsilon^{\beta x^2}g_\beta x$, $\gamma_\beta\left(x+\frac{\omega}{2}\right)=Ag\beta x$. It is scarcely necessary to remark that $\gamma_{-\beta}x$ has the same relations to the change of x into $x+\frac{\nu i}{2}$ as $\gamma_\beta x$ has to that of x into $x+\frac{\omega}{2}$.

VI.—ON BRIANCHON'S HEXAGON.

By Percival Frost, M.A., St. John's College.

The following is a proof of Brianchon's property of the hexagon circumscribing a conic section, in which the method of multiplication is used, which I made use of to solve a problem in transversals in No. xxi. p. 113. The property is stated in the last number, and proved for each case separately.

Let the conic sections be referred to two opposite sides of the circumscribing hexagon as axes, and let the equation to the curve be $ax+by-1=2B\sqrt{(xy)}$.

Then, if $ax+\beta y=1$ be the equation to a tangent,

$$ax+by-(ax+\beta y)=2B\sqrt{(xy)}$$

must give equal values to $\sqrt{\frac{y}{x}}$;

hence $$(a-\alpha)(b-\beta)=B^2 \ldots\ldots\ldots (1).$$

Let AB, CD (see fig. 2) be the two sides taken as axes, AE, EC and BF, FD pairs of contiguous sides.

Let the equations be

$$\left.\begin{aligned} ax + \beta' y &= 1 \quad \text{to } AE \\ a'x + \beta y &= 1 \quad \text{“ } EC \end{aligned}\right\} \ldots\ldots\ldots (2),$$

$$\left.\begin{aligned} \gamma x + \delta' y &= 1 \quad \text{“ } BF \\ \gamma' x + \delta y &= 1 \quad \text{“ } FD \end{aligned}\right\} \ldots\ldots\ldots (3),$$

multiply equations (2) by h and k respectively, and add; therefore at E we have the relation

$$(ha + ka')\,x + (h\beta' + k\beta)\,y = h + k.$$

Similarly at F we have the relation

$$(h'\gamma + k'\gamma')\,x + (h'\delta' + k'\delta)\,y = h' + k' \ldots\ldots (4);$$

and if the arbitrary multipliers be so chosen as to make these coincide, either will be the equation to the said line EF. In this case

$$\begin{aligned} ha + ka' &= h'\gamma + k'\gamma', \\ h\beta' + k\beta &= h'\delta' + k'\delta, \\ h + k &= h' + k'; \end{aligned}$$

therefore, eliminating by cross multiplication h and k,

$$\left.\begin{aligned} 0 = k' &\{\gamma'(\beta - \beta') + \delta(a - a') + a'\beta' - a\beta\} \\ + k' &\{\gamma(\beta - \beta') + \delta'(a - a') + a'\beta' - a\beta\} \end{aligned}\right\} \ldots (5).$$

Now by (1) we have the relations

$$(a - a).(b - \beta') = (a - a').(b - \beta) = (a - \gamma').(b - \delta) = \ldots$$

whence

$$\frac{\beta - \beta'}{a - a'} = \frac{b - \beta}{a - a} \ldots\ldots\ldots (6),$$

and

$$\frac{\delta - \beta'}{\gamma' - a} = \frac{\delta - b}{a - a};$$

therefore adding these equations we obtain

$$\frac{\gamma'(\beta - \beta') + \delta(a - a') + a'\beta' - a\beta}{(a - a')(\gamma' - a)} = \frac{\delta - \beta}{a - a}.$$

Similarly, by the symmetry of (1), changing γ and a and a into δ, β and b respectively, and *vice versâ*,

$$\frac{\delta'(a - a') + \gamma(\beta - \beta') + a'\beta' - a\beta}{(\beta - \beta')(\delta' - \beta)} = \frac{\gamma - a}{b - \beta};$$

therefore we have by (5) and (6)

$$0 = k'(\delta - \beta)(\gamma' - a) + h'(\delta' - \beta)(\gamma - a) \ldots\ldots (7).$$

Now $\alpha x + \delta y = 1$ is the equation to AD,
$\gamma x + \beta y = 1$ " " " BC;

therefore, multiplying by m and n, and adding,

$$(m\alpha + n\gamma)\, x + (m\delta + n\beta)\, y = m + n \ldots . \quad (8)$$

is a relation which holds at G, the intersection.

Assume m and n so as to satisfy the equations

$$m\alpha + n\gamma = h'\gamma + k'\gamma',$$
$$m\delta + n\beta = h'\delta' + k'\delta;$$

therefore $m(\gamma\delta - \alpha\beta) = h'(\gamma\delta' - \beta\gamma) + k'(\gamma\delta - \beta\gamma')$;

or $(m - k')(\gamma\delta - \alpha\beta) = h'\gamma(\delta' - \beta) + k'\beta(\alpha - \gamma')$.

Similarly $(n - h')(\gamma\delta - \alpha\beta) = h'\alpha(\beta - \delta') + k'\delta(\gamma' - \alpha)$;

hence $\{m + n - (h' + k')\}(\gamma\delta - \alpha\beta)$
$= h'(\gamma - \alpha)(\delta' - \beta) + k'(\delta - \beta)(\gamma' - \alpha) = 0$, by (7);

and $m + n = h' + k'$;

therefore the relations (8) and (4) coincide, or the point G is in the straight line EF; which proves the proposition.

VII.—ON THE LINES OF CURVATURE OF SURFACES OF THE SECOND ORDER.

By William Thomson, B.A. St. Peter's College.

The method usually followed in works on Geometry of Three Dimensions, in treating the differential equation to the lines of curvature of an ellipsoid, leads to an unsymmetrical integral, involving only two of the co-ordinates, and therefore representing the projection of the lines of curvature on one of the co-ordinate planes. In the following paper, by making use of an equivalent process, but preserving the symmetry with respect to the two variables which are involved, an integral is obtained which enables us from the symmetry to infer the equations to the projections on the other two co-ordinate planes. By combining the three forms of the integral thus obtained, we arrive at the integral given by Mr. Ellis, and at other symmetrical formulæ.

Let the equation to the surface be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

The differential equation to the lines of curvature is consequently $\dfrac{(b^2 - c^2)\, x}{dx} + \dfrac{(c^2 - a^2)\, y}{dy} + \dfrac{(a^2 - b^2)\, z}{dz} = 0.$

Let $\frac{x^2}{a^2} = u$, $\frac{y^2}{b^2} = v$, $\frac{z^2}{c^2} = w$. The preceding equations become

$$u + v + w = 1 \ldots\ldots\ldots\ldots (1),$$

$$\frac{(b^2 - c^2)\,u}{du} + \frac{(c^2 - a^2)\,v}{dv} + \frac{(a^2 - b^2)\,w}{dw} = 0 \ldots . (2).$$

Eliminating u from the latter equation, by means of the former, we have

$$\frac{b^2 - c^2}{du} + v\left(\frac{c^2 - a^2}{dv} - \frac{b^2 - c^2}{du}\right) + w\left(\frac{a^2 - b^2}{dw} - \frac{b^2 - c^2}{du}\right) = 0,$$

or
$$b^2 - c^2 = \frac{v\,(-c^2\,du - c^2\,dv + a^2\,du + b^2\,dv)}{dv} - \frac{w\,(-b^2\,du - b^2\,dw + a^2\,du + c^2\,dw)}{dw}.$$

But $du + dv + dw = 0$, by (1); and hence the equation to the lines of curvature may be put under the form

$$\frac{v}{dv} - \frac{w}{dw} = \frac{b^2 - c^2}{a^2\,du + b^2\,dv + c^2\,dw}.$$

If from this equation we eliminate du, we obtain an equation of Clairaut's form, of which the integral is found by substituting for $\frac{dv}{dw}$ an arbitrary constant. For the sake of symmetry we shall denote this constant by $\frac{g}{h}$; and we must consequently substitute g and h for dv and dw in the differential equation, and therefore also for du, $-(g + h)$, which we shall denote by f. Thus we have

$$\left.\begin{aligned} \frac{du}{f} = \frac{dv}{g} = \frac{dw}{h} \\ \text{where} \qquad f + g + h = 0, \end{aligned}\right\} \ldots\ldots\ldots (3),$$

and the complete integral is

$$\frac{v}{g} - \frac{w}{h} = \frac{b^2 - c^2}{a^2 f + b^2 g + c^2 h} \ldots\ldots\ldots (4).$$

Also, by the symmetry, we have for the integrals involving the variables wu and uv,

$$\left.\begin{aligned} \frac{w}{h} - \frac{u}{f} = \frac{c^2 - a^2}{a^2 f + b^2 g + c^2 h} \\ \frac{u}{f} - \frac{v}{g} = \frac{a^2 - b^2}{a^2 f + b^2 g + c^2 h} \end{aligned}\right\} \ldots\ldots\ldots (4),$$

The manner in which the quantities f, g, h have been introduced, shows clearly how they represent only one arbitrary constant. Any one of the equations (4) may be written in such a form as to contain only one arbitrary constant explicitly; and it will be shown below how f, g, h may be expressed symmetrically by two arbitrary constants, one of which is irrelevant, as it enters as a factor in the integral.

From the equations (4), as from the ordinary forms, the properties of the projections of the lines of curvature may be readily deduced. Thus, taking the second, and substituting for w, u, and g their values $\frac{z^2}{c^2}$, $\frac{x^2}{a^2}$, and $-(f+h)$, we have

$$\frac{z^2}{c^2 h} - \frac{x^2}{a^2 f} = \frac{a^2 - c^2}{(b^2 - c^2)h - (a^2 - b^2)f}.$$

Let a^2, b^2, c^2 be positive quantities in descending order of magnitude. Then, unless f and h have opposite signs, this equation cannot be satisfied by any values of z, x which satisfy the inequality

$$\frac{z^2}{c^2} + \frac{x^2}{a^2} < 1;$$

that is, by values which correspond to any point of the ellipsoid. Hence we may write the equation as

$$\frac{z^2}{\gamma^2} + \frac{x^2}{\alpha^2} = 1 \ldots\ldots\ldots\ldots\ldots (5),$$

where

$$\gamma^2 = \frac{c^2 (a^2 - c^2) h}{(b^2 - c^2)h - (a^2 - b^2)f},$$

$$\alpha^2 = -\frac{a^2 (a^2 - c^2) f}{(b^2 - c^2)h - (a^2 - b^2)f}.$$

Eliminating $f : h$ between these equations, we have

$$\frac{\gamma^2 (b^2 - c^2)}{c^2} + \frac{\alpha^2 (a^2 - b^2)}{a^2} = a^2 - c^2 \ldots\ldots (6).$$

Hence we conclude that the projections of the lines of curvature on the plane of the greatest and least axes of the ellipsoid, are the ellipses whose semiaxes, γ, α, are connected by the equation (6). Thus the construction for describing them is as follows. Draw an ellipse, concentric with the ellipsoid, in the plane ca, with the lines

$$c\left(\frac{a^2 - c^2}{b^2 - c^2}\right)^{\frac{1}{2}}, \quad a\left(\frac{a^2 - c^2}{a^2 - b^2}\right)^{\frac{1}{2}},$$

as semiaxes. Take any point in this ellipse, draw perpendiculars to the axes, and with the intersections as vertices,

describe a concentric ellipse. This will be the projection of a line of curvature. Also, by giving the point assumed in the auxiliary ellipse every possible position in its circumference, we obtain the projections of all the lines of curvature. Similar constructions are applicable to the projections of the lines of curvature on the other two principal planes; and, by taking one or two of the quantities a^2, b^2, c^2 negative, we may extend the rules to hyperboloids of one or of two sheets.

In the case we have taken, of an ellipsoid, and the plane of the greatest and least axes for the plane of projection, each curve intersects the consecutive one, and the locus of these intersections may be found from (5) and (6) by the ordinary process. Thus, by differentiation,

$$\frac{z^2}{\gamma^3}\,d\gamma + \frac{x^2}{a^3}\,da = 0,$$

$$\frac{a^2 - b^2}{a^2}\,a\,da + \frac{b^2 - c^2}{c^2}\,\gamma\,d\gamma = 0.$$

Hence
$$\frac{c^2}{b^2 - c^2}\,\frac{z^2}{\gamma^4} = \frac{a^2}{a^2 - b^2}\,\frac{x^2}{a^4},$$

which gives
$$\frac{\gamma^2}{cz\,(b^2 - c^2)^{-\frac{1}{2}}} = \frac{a^2}{ax\,(a^2 - b^2)^{-\frac{1}{2}}}.$$

By combining this equation, first with (6) and then with (5), we find each member

$$= \frac{a^2 - c^2}{\frac{z}{c}(b^2 - c^2)^{\frac{1}{2}} + \frac{x}{a}(a^2 - b^2)^{\frac{1}{2}}} = \frac{z}{c}(b^2 - c^2)^{\frac{1}{2}} + \frac{x}{a}(a^2 - b^2)^{\frac{1}{2}}.$$

Hence
$$\frac{z}{c}(b^2 - c^2)^{\frac{1}{2}} + \frac{x}{a}(a^2 - b^2)^{\frac{1}{2}} = (a^2 - c^2)^{\frac{1}{2}} \ldots\ldots (7),$$

is the equation to the required locus, which is therefore a group of four straight lines (on account of the double signs of the radicals) forming a rhombus, of which the diagonals coincide with the axes of c and a.

Thus we see that the projections of the lines of curvature on the plane of the greatest and least axes, are ellipses inscribed in a rhombus, with their axes coincident with those of the ellipsoid. If we consider b^2 as not of intermediate magnitude between a^2 and c^2, the equation (7) represents an imaginary group of straight lines, which shows that the projection of any line of curvature on the plane of the greatest and mean, or of the mean and least axes, does not meet its consecutive.

It may be remarked with respect to equations (4), that any one of them may be deduced from any other, by combining it with the equation $u + v + w = 1$, as is easily verified. Also, by multiplying the first by a^2, the second by b^2, and the third by c^2, and adding, we have

$$\frac{u(b^2 - c^2)}{f} + \frac{v(c^2 - a^2)}{g} + \frac{w(a^2 - b^2)}{h} = 0 \ldots\ldots (8),$$

which is the symmetrical integral given by Mr. Ellis, (Vol. II. p. 133). This equation might also have been found directly, when it was proved that

$$\frac{du}{f} = \frac{dv}{g} = \frac{dw}{h};$$

by eliminating by means of these relations, du, dv, dw from (2), the differential equation to the lines of curvature, which is the method followed by Mr. Ellis.

Without losing generality, we may substitute for f, g, h, any expressions in terms of two distinct constants which satisfy the condition $f + g + h = 0$. Thus, if we take k and υ for the constants, we may assume

$$f = ka^2(b^2 - c^2) - k\upsilon(b^2 - c^2),$$

or

$$\left.\begin{aligned} f &= k(b^2 - c^2)(a^2 - \upsilon), \\ g &= k(c^2 - a^2)(b^2 - \upsilon), \\ h &= k(a^2 - b^2)(c^2 - \upsilon), \end{aligned}\right\} \ldots\ldots\ldots (9).$$

Making these substitutions in (8), we have

$$\left.\begin{aligned} &\frac{u}{a^2 - \upsilon} + \frac{v}{b^2 - \upsilon} + \frac{w}{c^2 - \upsilon} = 0, \\ \text{or} \quad &\frac{x^2}{a^2(a^2 - \upsilon)} + \frac{y^2}{b^2(b^2 - \upsilon)} + \frac{z^2}{c^2(c^2 - \upsilon)} = 0, \end{aligned}\right\} \ldots\ldots (10).$$

Adding this equation, multiplied by υ, to

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \ldots\ldots\ldots (a),$$

we have

$$\frac{x^2}{a^2 - \upsilon} + \frac{y^2}{b^2 - \upsilon} + \frac{z^2}{c^2 - \upsilon} = 1 \ldots\ldots\ldots (11).$$

This equation shows that the lines of curvature of any surface of the second order with a centre, are its intersections with confocal surfaces of the second order. Since this property is independent of the centre, it follows that it must also be true for the case of a surface without a centre.

If the coordinates x, y, z, of any point in the line of curvature be given, by substituting these values for x, y, z

in (10), which will be a quadratic in v, we may determine two values of this parameter, which, substituted in (11), will give the equations to the hyperboloids of one sheet and of two sheets confocal with the ellipsoid, which cut it in the two lines of curvature passing through the given point x_1, y_1, z_1. In general, equation (11) may be considered as a cubic for determining v, when x, y, z have any given real values whatever, x_1, y_1, z_1. The three roots, which may readily be shown to be real, correspond to the three species of surfaces confocal with the ellipsoid (a, b, c) which intersect in the point x_1, y_1, z_1. In the present case, when this point is on the surface of the ellipsoid (a, b, c), and therefore x_1, y_1, z_1 satisfy the equation (a), one root of the cubic is zero, and the other two are the roots of the quadratic (10), which is the *reduced equation*. Thus, whether we take the form (10) or (11) of the integral, the two arbitrary constants may be determined by the solution of a quadratic equation, from the condition that the curve passes through a given point.

If we wish to determine the direction of the tangent at any point of a line of curvature, we may follow the usual process of differentiation for curves of double curvature.

Thus, taking any two of the equations (4), we find

$$\frac{du}{f} = \frac{dv}{g} = \frac{dw}{h}.$$

But if l, m, n be the direction-cosines of the tangent, we have

$$\frac{l}{\frac{a^2}{x}du} = \frac{m}{\frac{b^2}{y}dv} = \frac{n}{\frac{c^2}{z}dw};$$

and hence

$$\frac{l}{\frac{a^2}{x}f} = \frac{m}{\frac{b^2}{y}g} = \frac{n}{\frac{c^2}{z}h} \quad \ldots\ldots\ldots (12).$$

If we substitute for f, g, h their values by (9), these equations become

$$\frac{l}{\frac{a^2}{x}(b^2-c^2)(a^2-v)} = \frac{m}{\frac{b^2}{y}(c^2-a^2)(b^2-v)} = \frac{n}{\frac{c^2}{z}(a^2-b^2)(c^2-v)} \quad \ldots(13).$$

We may also determine l, m, n by the ordinary formulæ for the principal directions of curvature. Thus, if ρ be the radius of curvature of the normal section touching the line of curvature at the point considered, we have

$$l = \frac{\mu x}{a^2 - p\rho}, \quad m = \frac{\mu y}{b^2 - p\rho}, \quad n = \frac{\mu z}{c^2 - p\rho} \quad \ldots\ldots (14);$$

and $p\rho$ is one of the values of Q deduced from the quadratic equation

$$\frac{x^2}{a^2(a^2-Q)}+\frac{y^2}{b^2(b^2-Q)}+\frac{z^2}{c^2(c^2-Q)}=0 \ldots\ldots (15).$$

Also v is a root of this equation, since, when it is substituted for Q, the equation is identical with (10), one of the equations of the line of curvature considered. Thus a line of curvature is the locus of points on the surface, for which one root of (15) is constant. Now, by combining equations (13) and (14), we have

$$\left.\begin{aligned}&\frac{a^2(b^2-c^2)(a^2-v)(a^2-p\rho)}{x^2}\\ =&\frac{b^2(c^2-a^2)(b^2-v)(b^2-p\rho)}{y^2}\\ =&\frac{c^2(a^2-b^2)(c^2-v)(c^2-p\rho)}{z^2}\end{aligned}\right\} \ldots\ldots\ldots\ldots (16).$$

These equations show that v and $p\rho$ cannot both be constant, and therefore they must be different roots of the equation (15). Hence, if ρ' be the radius of curvature of a normal section perpendicular to the line of curvature at P, we have

$$v=p\rho';$$

which shows that the radii of curvature of sections perpendicular to a line of curvature at different points, are inversely proportional to the perpendiculars from the centre upon the tangent planes at those points.

The equations (16) may be verified directly, since, $p\rho$ and v being the two roots of (15), we have

$$v\,.\,p\rho=a^2b^2c^2\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}\right),$$

$$v+p\rho=\frac{(b^2+c^2)x^2}{a^2}+\frac{(c^2+a^2)y^2}{b^2}+\frac{(a^2+b^2)z^2}{c^2}.$$

Hence $(a^2-v)(a^2-p\rho)$

$$=a^4-\{a^2(b^2+c^2)-b^2c^2\}\frac{x^2}{a^2}-\{a^2(c^2+a^2)-c^2a^2\}\frac{y^2}{b^2}-\{a^2(a^2+b^2)-a^2b^2\}\frac{z^2}{c^2}$$

$$=a^4\left(1-\frac{y^2}{b^2}-\frac{z^2}{c^2}\right)-\{a^2(b^2+c^2)-b^2c^2\}\frac{x^2}{a^2}$$

$$=(a^2-b^2)(a^2-c^2)\frac{x^2}{a^2}.$$

Hence $\dfrac{a^2(b^2-c^2)(a^2-v)(a^2-p\rho)}{x^2}=-(b^2-c^2)(c^2-a^2))a^2-b^2);$

and therefore we infer that each member of (16) is equal to this expression, on account of the symmetry.

Let l', m', n' be the direction-cosines of the principal section corresponding to the root υ of the quadratic equation (15). We shall have, by the formulæ which correspond to (14),

$$l' = \frac{\mu x}{a^2 - \upsilon}, \quad m' = \frac{\mu y}{b^2 - \upsilon}, \quad n' = \frac{\mu z}{c^2 - \upsilon} \quad \ldots . \quad (17).$$

Thus, by means of the same root of the quadratic equation, we have, in (13) and (17), expressed the direction-cosines of each of the two principal sections. If we put λ for each member of (13), these equations give

$$ll' + mm' + nn' = \lambda\mu \{a^2(b^2 - c^2) + b^2(c^2 - a^2) + c^2(a^2 - b^2)\} = 0,$$

which proves that the principal directions of curvature are at right angles to one another; a theorem, of which many other different proofs have been given.

Dec. 1844.

VIII.—MATHEMATICAL NOTE.

1.—Light diverging from a point is incident on a given surface at a given point; find the direction of the reflected ray.

Let l, m, n be the direction cosines of the normal (N) at the point x, y, z,

α, β, γ } be those of the { incident ray (I),
α', β', γ' } { reflected ray (R),

ι the angle of incidence;

then, because (N), (I), (R) are in the same plane (f, g, h), we have

$$\left.\begin{aligned} fl + gm + hn &= 0, \\ f\alpha + g\beta + h\gamma &= 0, \\ f\alpha' + g\beta' + h\gamma' &= 0, \end{aligned}\right\} \ldots\ldots\ldots (1);$$

hence, λ, μ being indeterminate multipliers,

$$l = \lambda\alpha + \mu\alpha', \quad m = \lambda\beta + m\beta', \quad n = \lambda\gamma + \mu\gamma' \ldots . (2).$$

Also

$$\left.\begin{aligned} \cos\iota &= l\alpha + m\beta + n\gamma, \\ \cos\iota' &= l\alpha' + m\beta' + n\gamma', \\ \cos 2\iota &= \alpha\alpha' + \beta\beta' + \gamma\gamma', \end{aligned}\right\} \ldots\ldots\ldots (3);$$

hence, multiplying equations in (2) by l, m, n respectively, and adding, also by α, β, γ and α', β', γ', and subtracting the results, we get by (3)

$$1 = (\lambda + \mu)\cos\iota,$$

$$0 = \lambda - \mu,$$

or

$$\lambda = \mu = \frac{1}{2\cos\iota},$$

whence equations (2) give

$$\alpha' = 2l\cos\iota - \alpha,\quad \beta' = 2m\cos\iota - \beta,\quad \gamma' = 2n\cos\iota - \gamma \ldots (4),$$

or equations to the reflected ray (2) are

$$\frac{\xi - x}{2l\cos\iota - \alpha} = \frac{\eta - y}{2m\cos\iota - \beta} = \frac{\zeta - z}{2n\cos\iota - \gamma} \ldots\ldots (5).$$

Cor. 1. Let the incident ray pass through the origin; and let p be the length of the perpendicular from the origin on the tangent plane, r the distance of the point of incidence from the origin: then

$$\cos\iota = \frac{p}{r},\quad \frac{\alpha}{x} = \frac{\beta}{y} = \frac{\gamma}{z} = \frac{1}{r};$$

hence equations (5) become

$$\frac{\xi - x}{2lp - x} = \frac{\eta - y}{2mp - y} = \frac{\zeta - z}{2np - z} \ldots\ldots\ldots (6).$$

Ex. Let the surface be an ellipsoid, whose equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1;$$

then

$$l = \frac{px}{a^2},\quad m = \alpha,\quad n = \alpha,$$

and

$$\frac{\xi - x}{\left(2\frac{p^2}{a^2} - 1\right)x} = \frac{\eta - y}{\left(2\frac{p^2}{b^2} - 1\right)y} = \frac{\zeta - z}{\left(2\frac{p^2}{c^2} - 1\right)z} \ldots (7).$$

Cor. 2. When the reflected ray passes through the line,

$$\frac{\xi}{l_1} = \frac{\eta}{m_1} = \frac{\zeta}{n_1} = \rho \text{ suppose } \ldots\ldots\ldots (7),$$

we have, substituting in (6), and calling each fraction q,

$$\left.\begin{aligned} l_1\rho - (2lp - x)\,q &= x,\\ m_1\rho - (2mp - y)\,q &= y,\\ n_1\rho - (2np - z)\,q &= z, \end{aligned}\right\} \ldots\ldots\ldots (8);$$

and eliminating ρ and q, and substituting for l, m, n from the equation to the surface, we shall find the locus on which the rays (7) were before reflection.

Let λ_1 μ_1, ν_1 be multipliers; then putting

$$\left.\begin{aligned} \lambda_1 l_1 \quad + \quad \mu_1 m_1 \quad + \quad \nu_1 n_1 \quad &= 0,\\ \lambda_1(2lp - x) + \mu_1(2mp - y) + \nu_1(2np - z) &= 0, \end{aligned}\right\},$$

we get

$$\frac{\lambda_1}{2(m_1 n - m n_1)\,p - (m_1 z - n_1 y)} = \frac{\mu_1}{(\quad)} = \frac{\nu_1}{(\quad)};$$

hence, by (8),

$$(m_1 n - m n_1)\,x + (n_1 l - n l_1)\,y + (l_1 m - m l_1)\,z = 0;$$

or if $F(x, y, z) = 0$, be the equation to the surface, and

$$u = \frac{dF}{dx}, \quad v = \frac{dF}{dy}, \quad w = \frac{dF}{dz}, \quad \text{since } \frac{l}{u} = \frac{m}{v} = \frac{n}{w};$$

therefore the equation to the required locus is

$$(m_1 w - n_1 v)\, x + (n_1 u - l_1 w)\, y + (l_1 v - m_1 u)\, z = 0 \ldots (9).$$

Ex. In the ellipsoid this becomes

$$\left(\frac{m_1 z}{c^2} - \frac{n_1 y}{b^2}\right) x + \left(\frac{n_1 x}{a^2} - \frac{l_1 z}{c^2}\right) y + \left(\frac{l_1 y}{b^2} - \frac{m_1 x}{a^2}\right) z = 0,$$

or $$l_1 yz\left(\frac{1}{b^2} - \frac{1}{c^2}\right) + m_1 zx\left(\frac{1}{c^2} - \frac{1}{a^2}\right) + n_1 xy\left(\frac{1}{a^2} - \frac{1}{b^2}\right) = 0,$$

the equation to a cone.

If in particular $l_1 = m_1 = n_1$, this reduces to

$$yz\left(\frac{1}{b^2} - \frac{1}{c^2}\right) + zx\left(\frac{1}{c^2} - \frac{1}{a^2}\right) + xy\left(\frac{1}{a^2} - \frac{1}{b^2}\right).$$

Addition.

For *refraction*, when ι' is the angle of refraction, and $\sin \iota = \kappa \sin \iota'$, we get equations (1), (2) unaltered; instead of (3) we have

$$\cos \iota = l\alpha + m\beta + n\gamma, \quad \cos \iota' = l\alpha' + m\beta' + n\gamma',$$
$$\cos (\iota - \iota') = \alpha\alpha' + \beta\beta' + \gamma\gamma';$$

whence it easily follows that

$$-\frac{\lambda}{\sin \iota'} = \frac{\mu}{\sin \iota} = \frac{1}{\sin (\iota - \iota')},$$

or $$l \sin (\iota - \iota') = -\alpha \sin \iota' + \alpha' \sin \iota;$$

and hence equations to the refracted ray

$$\frac{\xi - x}{l \sin (\iota - \iota') + \alpha \sin \iota'} = \frac{\eta - y}{m \sin (\iota - \iota') + \beta \sin \iota'}$$
$$= \frac{\zeta - z}{n \sin (\iota - \iota') + \gamma \sin \iota'} \ldots . (5').$$

These expressions give those for reflection (5) by putting $\kappa = -\iota$; $\iota = -\iota'$, as it ought to be.

L. Fischer.

END OF VOL. IV.

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